Applications of the Integral

(Area, Volume, & Work)

(Notes by Michael Samra)

The applications of the integral presented below are the area bounded by two curves, the volume of solids of revolution, and work. The approach taken is to determine the "infinitesimal element" that the integral adds up to produce the required quantity. For the area under the graph of a function, the infinitesimal area element is a (vertical) rectangle with height $f(x)$ and "width" or thickness dx - that is, $dA = f(x)\,dx$. In some instances, rather than adding vertical "slices", it will be more useful to use horizontal slices, which have thickness dy. The variable of integration is then y.

The examples below usually follow this procedure:

1) choose a vertical or horizontal slice.
2) determine the infinitesimal element.
3) determine the limits of integration.
4) calculate the integral.

Area

To find the area bounded by two curves, first solve for their points of intersection. (There's often more than one way of doing this). Next, at least roughly sketch their graphs. If a vertical slice is chosen, the thickness of the rectangle is dx and the height is the difference of the y-values of the two functions. A horizontal slice yields a rectangle on its side with thickness dy, and length the x-coordinate of the rightmost curve minus the x-coordinate of the other curve, where the x-coordinates must be written in terms of the variable of integration y. The second example illustrates this case.

Example: Find the area bounded by the curves

$$x - 2y = 0, \; 3y - 3x + x^2 = 0.$$

To find the points of intersection, substitute $y = \frac{x}{2}$ into the second equation to obtain:

$$\frac{3}{2}x - 3x + x^2 = 0$$

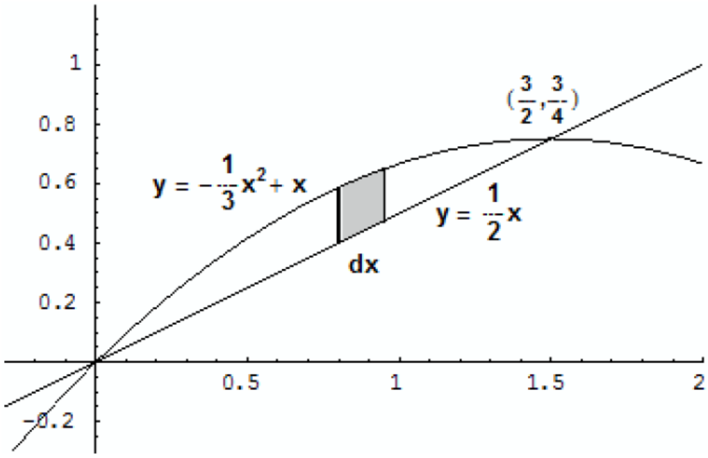
$$x^2 - \frac{3}{2}x = 0$$

$$x\left(x - \frac{3}{2}\right) = 0$$

$$x = 0 \text{ or } x = \frac{3}{2}$$

$$y = 0 \qquad y = \frac{3}{4}$$

(see the graph below)



An area element has height the y-coordinate of the parabola minus the y-coordinate of the line, times the width d x. The difference in y-coordinates in terms of the variable of integration x is $\left(-\frac{1}{3} x^2 + x\right) - \left(\frac{1}{2} x\right) = -\frac{1}{3} x^2 + \frac{1}{2} x$. Hence, the area element is $\left(-\frac{1}{3} x^2 + \frac{1}{2} x\right) d x$, and the integral ranges from $x = 0$ to $x = \frac{3}{2}$:

$$\int_0^{\frac{3}{2}} \left(-\frac{1}{3} x^2 + \frac{x}{2}\right) d x = \left(-\frac{x^3}{9} + \frac{x^2}{4}\right)\Big]_0^{\frac{3}{2}} = -\frac{3}{8} + \frac{9}{16} = \frac{3}{16}.$$

Example: Find the area bounded by the curves

$$x - 2 = y^2, \quad x + y = 4.$$

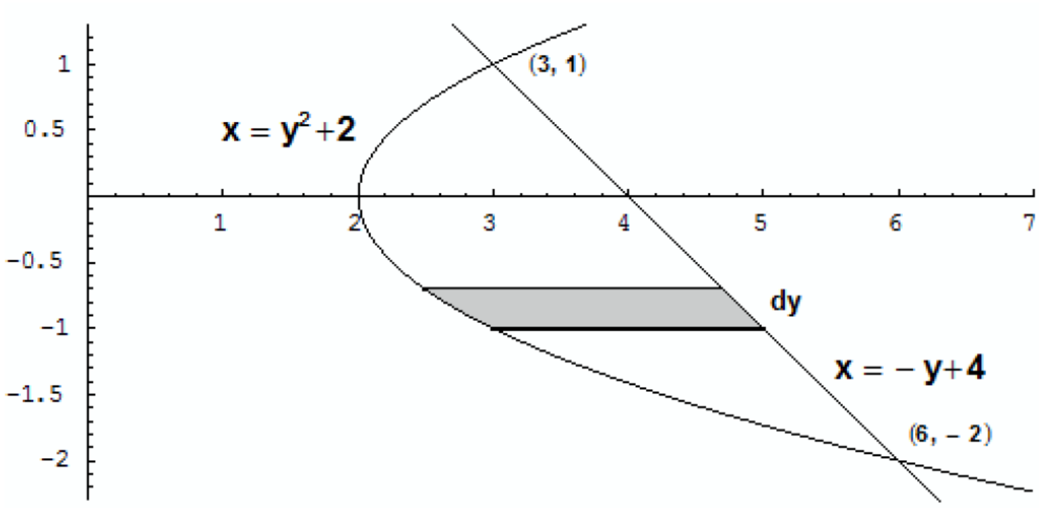
Substitute $x = y^2 + 2$ into the second equation, to obtain the quadratic equation:

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$y = -2$, or $y = 1$

$x = 6$ $\quad$ $x = 3$ $\quad$ (see the graph below).



In this case, it is easier to integrate with respect to the variable y. Otherwise the integral would have to be split into two parts, one part of which would involve using the two square roots of $(x - 2)$.

The area element in this case is the x-coordinate of $x + y = 4$ minus the x-coordinate of $x - 2 = y^2$, times the width d y. Since the integral is in terms of the variable y, the area element is:

$[(-y + 4) - (y^2 + 2)]\, d y = (-y^2 - y + 2)\, d y$, and y ranges from -2 to 1.

$$\int_{-2}^{1} (-y^2 - y + 2)\, d y = \left(-\frac{y^3}{3} - \frac{y^2}{2} + 2 y\right)\Big]_{-2}^{1} = \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right)$$
$$= \frac{7}{6} - \left(-\frac{10}{3}\right) = \frac{27}{6}.$$

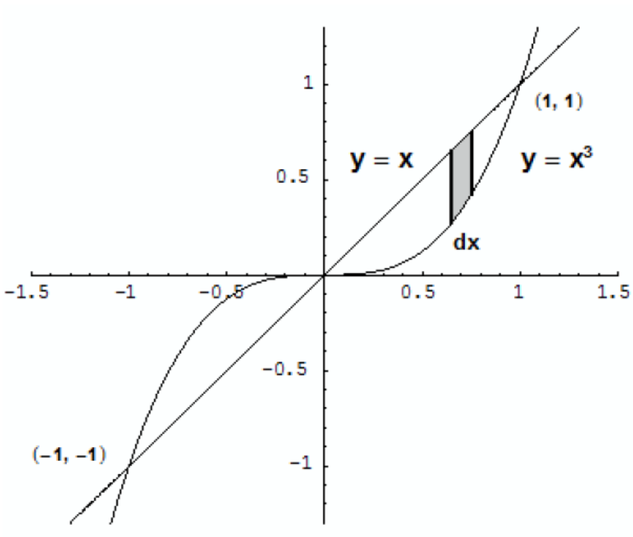
Example: Find the area bounded by the graphs of the functions

$$y = x^3, \quad y = x.$$

First, the points of intersection are found:

$$x^3 = x$$
$$x^3 - x = 0, \quad x(x^2 - 1) = 0$$
$$x(x + 1)(x - 1) = 0.$$

There are three points of intersection: $(-1, -1)$, $(0, 0)$, and $(1, 1)$. The region is given below:

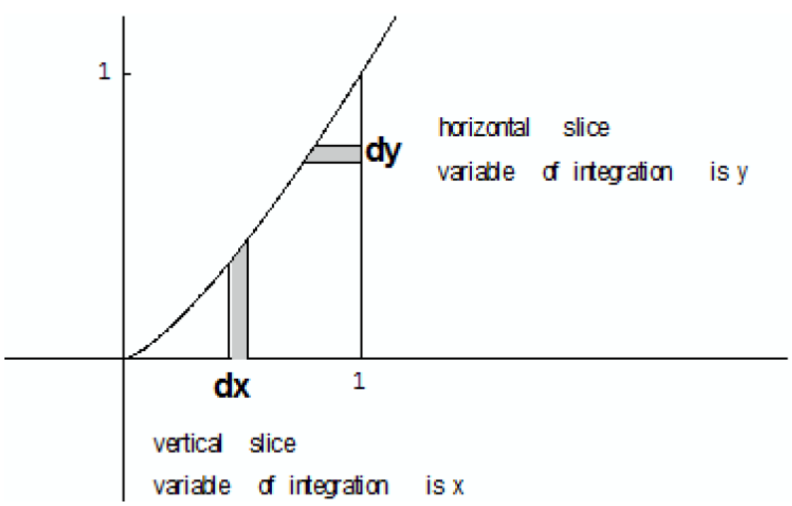


It should be evident by the symmetry of the graph that the area of the two regions can be found by finding the area between $x = 0$ and $x = 1$, and then multiplying by 2:

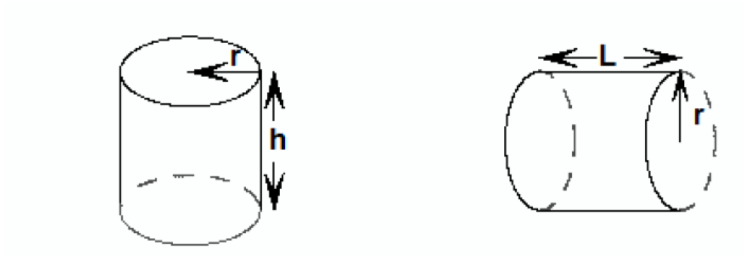
$$2\int_0^1 (x - x^3)\,dx = 2\left(x^2 - \frac{x^4}{2}\right)\Big]_0^1 = 2\left(1 - \frac{1}{2}\right) = 1.$$

Volume of Solids of Revolution

The methods of calculating volumes of solids generated by rotating curves about the x-axis, the y-axis, or another line are called the disk (or washer) method, and the method of cylindrical shells. Rather than memorizing the formulas for these methods it is easier to determine the volume element for each specific problem. The volume element is obtained be rotating an area element - an infinitesimal vertical or horizontal rectangle inside the bounded area - about a given axis or line. A vertical slice means that the volume element has to be written in terms of the variable of integration x, and for a horizontal slice, in terms of y.



The volume element will take the form of area $\cdot$ d x (or area $\cdot$ d y), where the area formulas that are used are the area of a circle (in which case the volume element will be a disk), and the surface area of a cylinder open at both ends (in which case the volume element will be a cylindrical shell). It's easy to remember the formula for the surface area of a cylinder as the circumference times the height or the circumference times the length.



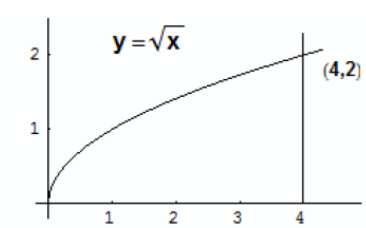
$A = 2\pi r h$ or $A = 2\pi r L$

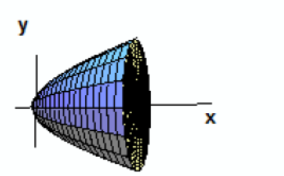
The examples below will clarify the procedure.

Example: Find the volume of the solid generated by rotating the area bounded by the curve $y = \sqrt{x}$ and the line $x = 4$ about a) the x-axis and b) the y-axis.

a) about the x-axis

First, graph the area that is to be revolved about the x-axis to produce the solid:

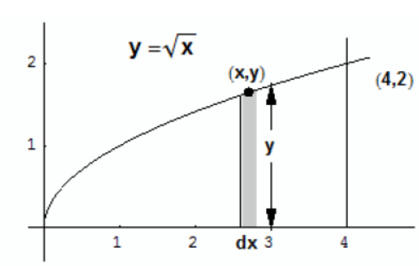


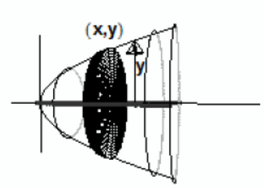


graph of the region to be rotated about the x-axis

the solid of revolution that is generated

If a vertical slice is used, then the volume element will be a disk, as pictured below:





area element to be rotated about the x-axis

volume element

The volume element is the area of the face of the disk, πr^2, times the thickness of the disk, dx. From the picture, you can see that the radius r is the height y of the disk, but since the variable of integration is x, y is replaced by $\sqrt{x}$:

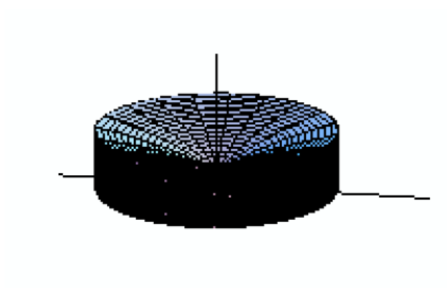
$$\text{volume element: } \pi\left(\sqrt{x}\right)^2 dx = \pi\, x\, dx$$

The next step is to determine the limits of integration. Here, the volume elements are added up from $x = 0$ to $x = 4$, and so the integral that calculates the volume is:

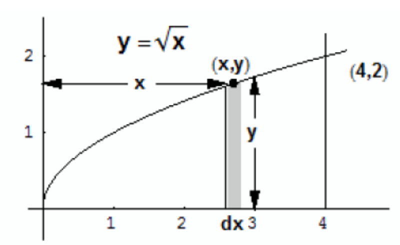
$$\int_0^4 \pi\, x\, dx = \pi \frac{x^2}{2} \Big]_0^4 = 8\,\pi.$$

b) about the y-axis.

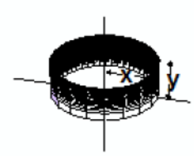
The solid generated by rotating the bounded area about the y-axis is pictured below:



The volume element is a circular wall called a cylindrical shell:



area element to be rotated about the y-axis



volume element

When the cylindrical shell method is used, it is especially important to keep in mind what the variable of integration is. The volume element in this example is given by the surface area of the cylinder, $2\,\pi\, r\, h$, times the thickness, dx. The radius r is simply x, whereas the height is y. Since x is the variable of integration, the height y must be written in terms of x, so y is replaced with $\sqrt{x}$:

$$\text{volume element: } 2\,\pi\, x \sqrt{x}\; dx = 2\,\pi\, x^{\frac{3}{2}}\, dx$$

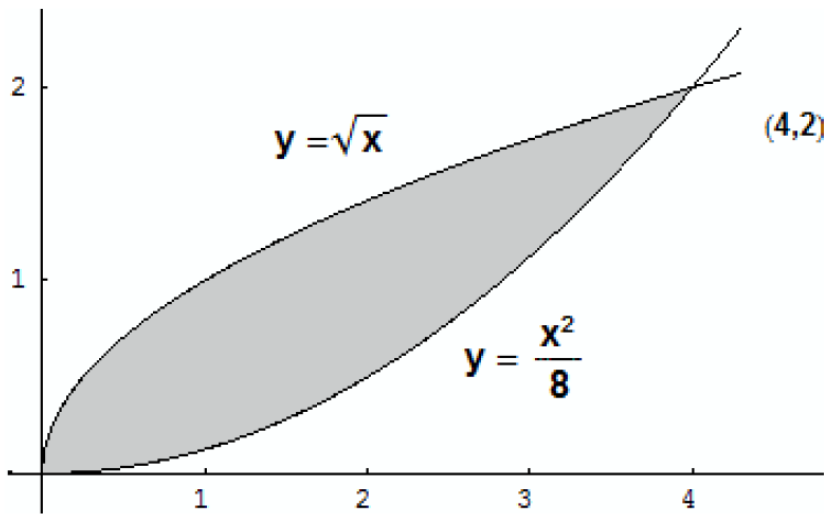
Again, x varies from 0 to 4, and so the volume is:

$$\int_0^4 2\,\pi\, x^{\frac{3}{2}}\, dx = \frac{4}{5}\,\pi\, x^{\frac{5}{2}} \Big]_0^4 = \frac{4}{5}\,\pi\left(4^{\frac{5}{2}}\right) = \frac{128\,\pi}{5}.$$

The disk (or washer) method is used when the rectangular slice is perpendicular to the axis of revolution, and the method of cylindrical shells is used when the rectangular slice is parallel to the axis of revolution.

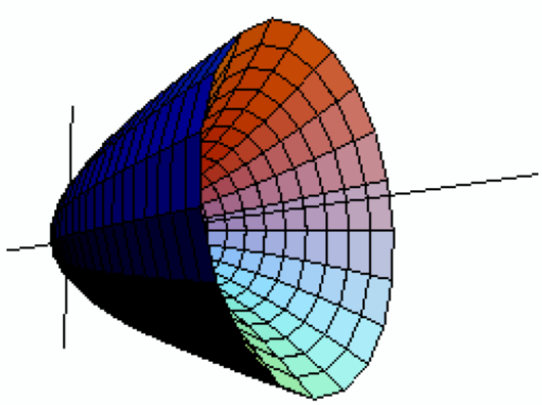
Example: Find the volume of the solid generated by revolving the area bounded by the curves $y = \sqrt{x}$ and $y = \frac{x^2}{8}$ about a) the x-axis, using the washer method; b) the y-axis, using the washer method; and c) the line $y = -1$, using the shell method.

The first step is to find the points of intersection of the two equations, and then to graph the bounded area. You can easily find that $y = \frac{x^2}{8}$ lies below the graph of $y = \sqrt{x}$ by comparing values of these curves between the points of intersection.

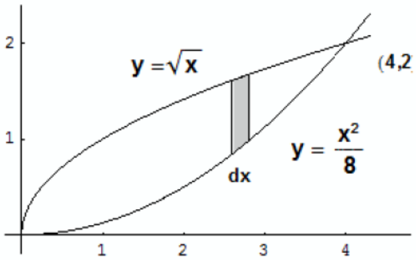


a) the x - axis, using the washer method.

The solid is first shown below:



To use the washer method, a vertical slice must be used. The volume element, called a washer, is pictured below:



area element to be rotated about the x-axis

volume element

The volume element will be the area of the entire circle minus the area of the circle that is the hole of the washer, times the thickness dx. The radius of the outer circle is given by the y-coordinate of the curve $y = \sqrt{x}$, and the radius of the inner circle is given by the y-coordinate of the curve $y = \frac{x^2}{8}$. Since the variable of integration is x, the volume element is given by:

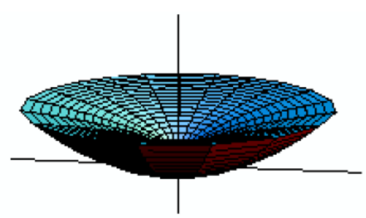
$$\text{volume element: } \left(\pi\left(\sqrt{x}\right)^2 - \pi\left(\frac{x^2}{8}\right)^2\right) dx = \left(\pi x - \pi \frac{x^4}{64}\right) dx$$

Note that this is not the same as $\pi\left(\sqrt{x} - \frac{x^2}{8}\right)^2 dx$. The integral is calculated from $x = 0$ to $x = 4$:

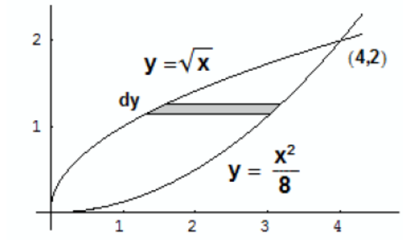
$$\int_0^4 \left(\pi x - \pi \frac{x^4}{64}\right) dx = \pi \int_0^4 x\, dx - \frac{\pi}{64} \int_0^4 x^4\, dx = \frac{24\pi}{5}.$$

b) the y-axis, using the washer method.

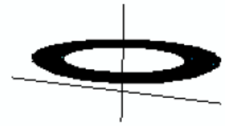
The solid is shown below:



To use the washer method, a horizontal slice must be used (since it is perpendicular to the y-axis). The volume element is pictured below:



area element to be rotated about the y-axis



volume element

The radius of the outer circle is given by the x-coordinate of the curve $y = \frac{x^2}{8}$, and since y is the variable of integration, x must be found in terms of y : $x = \sqrt{8\,y}$. The radius of the inner circle is given by the x- coordinate of the curve $y = \sqrt{x}$, so $x = y^2$.

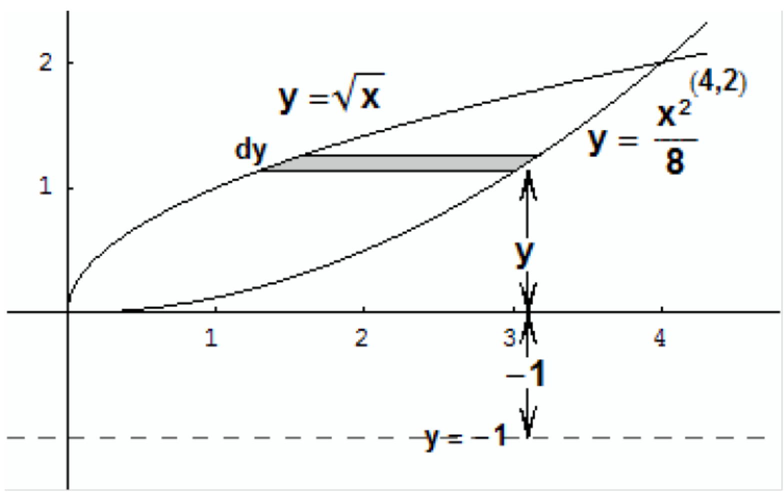
volume element: $(8\,\pi\,y - \pi\,y^4)\,d\,y$.

Next, the limits of integration must be found. Here, y varies from 0 to 2, so the volume is:

$$\int_0^2 (8\,\pi\,y - \pi\,y^4)\,d\,y = \left(4\,\pi\,y^2 - \frac{\pi\,y^5}{5}\right)\Big]_0^2 = \frac{48\,\pi}{5}.$$

c) the line $y = -1$, using cylindrical shells.

Since the shell method is being used, a horizontal slice must be used:



The radius of the cylinder is given by $y + 1$, and the length of the cylinder is given by the x-coordinate of the curve $y = \frac{x^2}{8}$ minus the x-coordinate of the curve $y = \sqrt{x}$. Since the variable of integration is y, the length of the cylinder is $\sqrt{8\,y} - y^2$.

volume element: $2\,\pi(y + 1)\left(\sqrt{8\,y} - y^2\right)d\,y$.

Again, the shells are added up from $y = 0$ to $y = 2$, and the integral is:

$$\int_0^2 2\,\pi(y + 1)\left(\sqrt{8\,y} - y^2\right)d\,y$$

$$= 2\,\pi \int_0^2 \left(8^{\frac{1}{2}}\,y^{\frac{3}{2}} - y^3 + 8^{\frac{1}{2}}\,y^{\frac{1}{2}} - y^2\right)d\,y = \frac{152\,\pi}{15}.$$

Work

Work measures the energy a force uses when displacing an object. The work done by a constant force to move an object a given distance in the direction of the force is defined as:

$$\text{Work} = \text{force} \cdot \text{distance}$$

For example, if a constant force of 1000 pounds moves a car 50 feet, then the work done is 50,000 foot- pounds. For the work required to lift a 50 kilogram bag 2 meters, the mass of 50 kilograms first has to be converted into the units of weight (force) according to Newton's equation, $F = m\,a = m\,g$, where $g = 9.8\,\frac{\text{meters}}{\text{sec}^2}$ is the acceleration due to gravity. The work done is then $(9.8)(50)(2) = 980$ newton-meters (or joules).

If the force isn't constant along the displacement [a, b] of the object, then work is defined to be the average value of f on [a, b] times the distance $(b - a)$, that is

$$W = \frac{1}{b-a}\int_a^b f(x)\,dx \cdot (b-a) = \int_a^b f(x)\,dx$$

A good example is provided by springs, where the force required to hold the spring in place increases the further it is stretched. This force is governed by Hooke's Law, that states the force is proportional to the number of units x that a spring is stretched beyond it's natural length, that is

$$f(x) = k\,x$$

where k is the spring constant. k is usually found by knowing the value of f for some position x.

Example: A force of 15 lbs is required to maintain a spring 5 feet beyond its natural length. What is the work required to pull the spring 3 inches beyond its natural length?

$15 = k \cdot 5$, so $k = 3$.

$$W = \int_0^{\frac{1}{4}} 3\,x\,dx = \frac{3\,x^2}{2}\Big]_0^{\frac{1}{4}} = \frac{3}{32} \text{ foot-pounds, (where 3 inches } = \tfrac{1}{4}\text{ feet).}$$

More complicated examples occur when trying to find the work done in pumping a liquid out of a tank. Here both force and distance can vary with the level x (or y) of the liquid that is being pumped out of the tank. For these problems, it's best to first determine the infinitesimal work element d W - the work done in lifting an infinitesimal volume element at an arbitrary level x of the liquid - and then integrate along the levels of the water that are being removed. More specifically, at an arbitrary level of the liquid x, d W will have the form

(weight of volume element) · (distance lifted) = (weight per unit volume · $A(x)\,d\,x$) · $s(x)$,

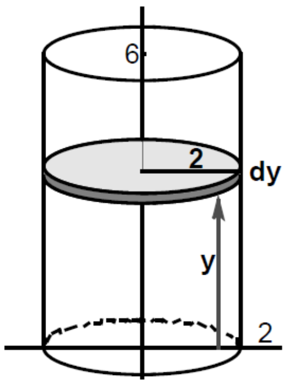
where $A(x)\,d\,x$ = the volume element that is being lifted, weight per unit volume · $A(x)\,d\,x$ = the weight of the volume element, and $s(x)$ = the distance the volume element at level x must be lifted.

There are often more than one way to usefully set up coordinates for these problems.

Example: A cylindrical tank of radius 2 feet and height 6 feet is full of water. The weight per unit volume of water is 62.5 $\frac{\text{lbs}}{\text{ft}^3}$.

a) Find the work done in pumping the water to the top of the tank.
b) Find the work done in pumping the water to an outlet 2 feet above the top of the tank.
c) Now suppose the tank is lying on its side. Find the work done in pumping the water to the top of the tank. (Do you expect this number to be less than, equal to, or greater than the number you found in (a))?

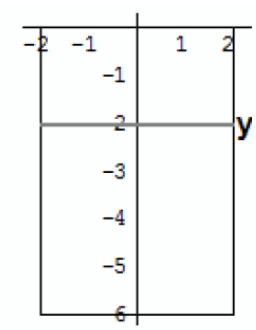
a) In this case, the volume elements are disks with constant radius 2 and thickness d y.

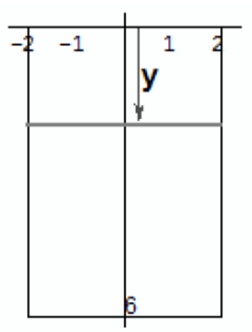


At level y the volume element must be lifted a distance $(6 - y)$ feet, so altogether the infinitesimal work element is $(62.5)\, 4\,\pi(6 - y)\, d\, y$, with the integral varying from 0 to 6:

$$250\,\pi \int_0^6 (6 - y)\, d\, y = \left(1500\,\pi\, y - 250\,\pi \frac{y^2}{2}\right)\Big]_0^6 = 4500\,\pi \text{ foot-pounds.}$$

Two other coordinatizations, with the accompanying integral, are given below:





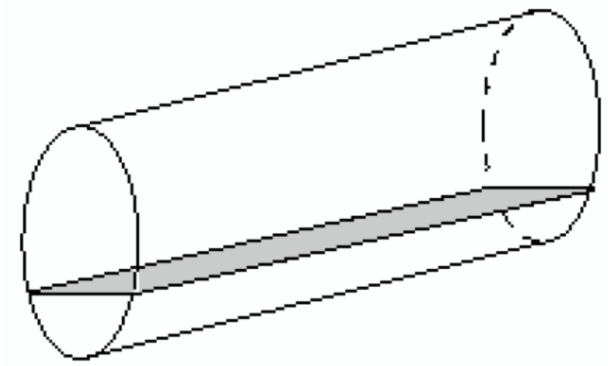
$(62.5)\,(4\,\pi) \int_{-6}^{0} (-y)\, d\, y$

$(62.5)\,(4\,\pi) \int_0^6 y\, d\, y$ (down is considered the positive direction)

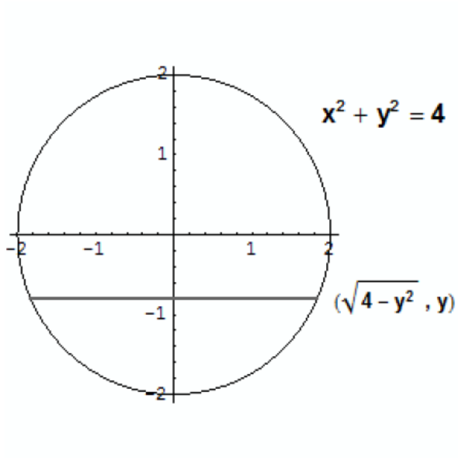
b) Here, the distance function is replaced by $(8 - y)$, and the integral is

$$(62.5)\,(4\,\pi) \int_0^6 (8 - y)\, d\, y = 2000\,\pi\, y - 250\,\pi \frac{y^2}{2} \Big]_0^6 = 7500\,\pi \text{ foot-pounds.}$$

c) The cylinder has radius 2 feet and length 6 feet. On its side, horizontal (level) slices are rectangles with constant length 6 feet but variable width.



To find the width at an arbitrary level y, a coordinate system can be set up as below:



The width at level y is $2\sqrt{4-y^2}$, the length is 6 feet, and the volume element must be lifted $(2 - y)$ feet. The infinitesimal work element is $(62.5)(6)\,2\sqrt{4-y^2}\,(2-y)\,dy$, and the integral is evaluated from $y = -2$ to $y = 2$:

$$(62.5)(6)(2)\int_{-2}^{2}\sqrt{4-y^2}\,(2-y)\,dy = 750\left(2\int_{-2}^{2}\sqrt{4-y^2}\,dy - \int_{-2}^{2}y\sqrt{4-y^2}\,dy\right).$$

The first integral represents the area of a circle centered at the origin, and so its value is 4π. The second integral can be done using a u-substitution, but one can also notice that the integrand is odd (the values are opposite on either side of the y-axis), and since the integral is done with 0 in the middle of the interval of integration, the integral has value 0. Altogether, the answer is

$$(750)(4\pi) = 3000\pi \text{ foot-pounds.}$$

It should be expected that the work required to pump out the water is greater for the vertical cylindrical tank than the horizontal cylindrical tank because, on average, the same volume of water has to be lifted a greater distance for the vertical tank.

If the kilogram-meter-second system is used, the volume element has to be first multiplied by the density of the liquid (the mass per unit volume), and then by $9.8\ \frac{\text{meters}}{\text{sec}^2}$ to obtain the weight per unit volume. For example, for water, the volume element has to be multiplied by the density of water $(1000\ \frac{\text{kg}}{\text{m}^3})$, and then by 9.8 to obtain the weight of the volume element.