Differentiation

(Notes by Michael Samra)

The first property of functions that is learnt in Calculus is continuity. Continuity is defined using a limit: a function $f(x)$ is said to be *continuous* at a number (or point) a if $\lim_{x\to a} f(x) = f(a)$. In words, $f(x)$ is continuous at a if the limit of $f(x)$ as x approaches a is equal to the value of f at a. Examples of functions that are continuous at every point in their domain are polynomial functions, rational functions, root functions, and trigonometric functions.

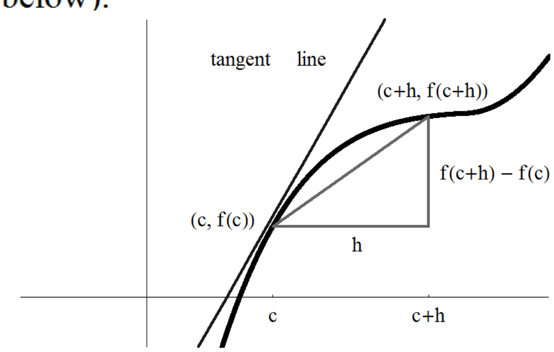
Continuity distinguishes those functions which do not "skip" values. The Intermediate Value Theorem guarantees, for example, that if a function is continuous on a closed interval $[a, b]$ and is negative at one endpoint and positive at the other endpoint then there must be a point between a and b where the function is equal to 0. This is used to find solutions to equations.

Differentiability is also defined using a limit, but by a different limit than the limit of a function at a point. Differentiability at a point is defined by finding if the limit of what's called the *difference quotient* (or *Newton Quotient*) of a function exists at that point.

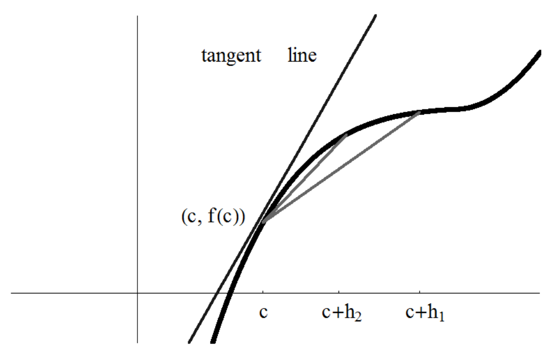
$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \text{ exists.}$$

If this limit exists, it is called the *derivative* of f at c and is denoted by $f'(c)$.

The difference quotient $\frac{f(c+h)-f(c)}{h}$ is similar to the expression $\frac{\Delta y}{\Delta x} = \frac{y_2-y_1}{x_2-x_2}$ that is used to find the slope of a line ($y_2 = f(c+h)$, $y_1 = f(c)$, $x_2 = c+h$, $x_1 = c$). Graphically, $\frac{f(c+h)-f(c)}{h}$ is the slope of the (secant) line connecting the points $(c, f(c))$ and $(c+h, f(c+h))$ on the graph of the function. In the limit, the number $f'(c)$ is the slope of the line tangent to the graph of f at $(c, f(c))$ (see the graphs below).



The secant line is the hypotenuse of the triangle above.



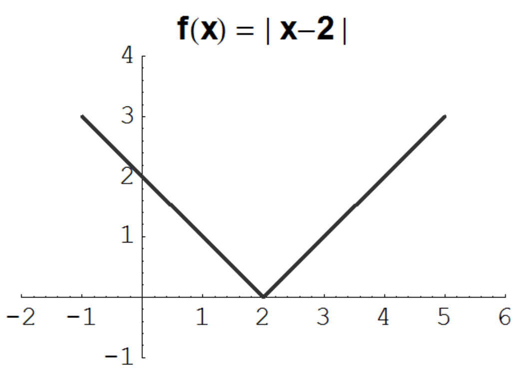
As h gets smaller, the slope of the secant line approaches the slope of the tangent line.

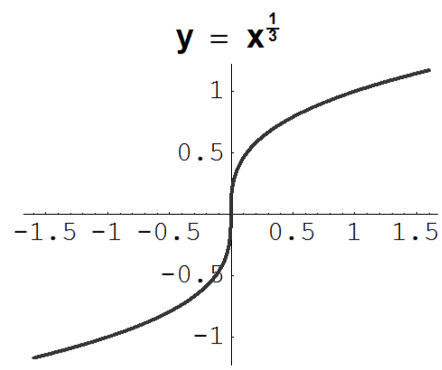
For a car traveling along a straight road whose position at time t is given by the function $s(t)$, the difference quotient $\frac{s(t+h)-s(t)}{(t+h)-t} = \frac{s(t+h)-s(t)}{h}$ is the *average velocity* of the car over the interval $[t, t+h]$. This is similar to the formula rate = $\frac{\text{distance}}{\text{time}}$ that applies to objects that travel with constant velocity. To define the *instantaneous velocity* or simply the *velocity* $v(t)$ of the car at time t, the limit is taken as h approaches 0. That is, the velocity of the car at any time t is the limit of the average velocities of the car over smaller and smaller intervals.

More generally, the derivative of a function $f(x)$ at some point measures the *rate of change* of the function with respect to the independent variable x at that point.

It's more difficult for a function to satisfy the differentiability condition than the continuity condition: it can be shown that if a function is differentiable at a point c then the function is continuous at c.

The converse, however, is not true. Below are examples of functions that are continuous at a certain point, but not differentiable at that point.





For the absolute value function on the left, if you calculate the difference quotient at $c = 2$ as h approaches 0 from the left, you find: $\lim_{h\to 0^-} \frac{|2+h-2|-|2-2|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$. ($|h| = -h$ since h is negative), whereas the limit from the right is the slope of the line to the right of $c = 2$, which is 1. Since the limit from the left is not equal to the limit from the right, the limit does not exist at $c = 2$. For the second function (the inverse to $y = x^3$), the tangent line to the graph at (0,0) is vertical, so the limit of the difference quotient at $c = 0$ approaches infinity.

The properties of differentiability and continuity are also different in this respect: a function f that is differentiable at each point of its domain generates a new function, called the derivative f'. This new function provides information about the original function. For example, wherever $f' > 0$, f is increasing.

To <u>differentiate</u> a function $f(x)$ is to find its derivative. When using the limit of the difference quotient to find the equation for this new function, the number c is replaced with x.

Example: Use the definition of the derivative to find $f'(2)$, where $f(x) = x^3 - 1$.

$$f'(2) = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3-1-(2^3-1)}{h} = \lim_{h\to 0} \frac{(8+12\,h+6\,h^2+h^3)-1-(7)}{h}$$

$$= \lim_{h\to 0} \frac{8+12\,h+6\,h^2+h^3-8}{h} = \lim_{h\to 0} \frac{h(12+6\,h+h^2)}{h} = \lim_{h\to 0} \left(12+6\,h+h^2\right) = 12$$

Example: Use the definition of the derivative to find $f'(x)$, where $f(x) = \sqrt{x+4}$.

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h+4}-\sqrt{x+4}}{h} = \lim_{h\to 0} \frac{\sqrt{x+h+4}-\sqrt{x+4}}{h} \cdot \frac{\sqrt{x+h+4}+\sqrt{x+4}}{\sqrt{x+h+4}+\sqrt{x+4}}$$

$$= \lim_{h\to 0} \frac{(x+h+4)-(x+4)}{h(\sqrt{x+h+4}+\sqrt{x+4})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+4}+\sqrt{x+4})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+4}+\sqrt{x+4}}$$

$$= \frac{1}{2\sqrt{x+4}}.$$

Notation:

There are several ways of asking to find the derivative of a function $f(x)$:

$\frac{d}{dx} f(x)$ $\qquad D_x\, f(x)$ $\qquad (f(x))'$

Differentiation Formulas

The derivatives of some basic functions that are given here, along with the differentiation rules for functions given below, enable one to differentiate many functions without having to calculate the limit of the difference quotient.

The differentiation formula, $\frac{d}{dx} x^n = n\,x^{n-1}$, called the Power Rule, is first proved for n a positive integer using the fact that $\frac{d}{dx} k = 0$ (k a constant), $\frac{d}{dx} x = 1$, and a factoring formula. The rule is shown to hold for negative integer exponents by applying the Quotient Rule given below for differentiating quotients of functions. Using the method of implicit differentiation (discussed later in these notes), the Power Rule is shown to hold for rational exponents. Finally, in the second semester of Calculus, irrational exponents are defined, and it is shown that the Power Rule holds for any real exponent.

Power Rule: $\frac{d}{dx} x^r = r\,x^{r-1}$, r any real number.

Example: $\frac{d}{dx} x^4 = 4x^3$.

The derivatives of the trigonometric functions should be memorized:

$$\frac{d}{dx} \sin x = \cos x \qquad \frac{d}{dx} \cos x = -\sin x$$
$$\frac{d}{dx} \tan x = \sec^2 x \qquad \frac{d}{dx} \sec x = \sec x \cdot \tan x$$
$$\frac{d}{dx} \cot x = -\csc^2 x \qquad \frac{d}{dx} \csc x = -\csc x \cdot \cot x$$

The bottom four are derived from the top two by applying the Quotient Rule. It might be worth remembering that the derivatives of the trigonometric functions have the same groupings as the Pythagorean identities ($\sin x$ and $\cos x$, $\tan x$ and $\sec x$, $\cot x$ and $\csc x$).

Differentiation Rules

For each operation involving functions, there is a corresponding rule for differentiation. The rules for differentiating a constant times a function, and for differentiating the sum or difference of two functions are especially easy:

1. $(k\,f(x))' = k\,f'(x)$, k any constant.
2. $(f(x) + g(x))' = f'(x) + g'(x)$.
3. $(f(x) - g(x))' = f'(x) - g'(x)$.

Example: Find $\frac{d}{dx}\left(2x^3 + \sec x\right)$.

$$\frac{d}{dx}\left(2x^3 + \sec x\right) = \frac{d}{dx} 2x^3 + \frac{d}{dx} \sec x = 2\frac{d}{dx} x^3 + \frac{d}{dx} \sec x = 6x^2 + \sec x \tan x$$

The rules for the derivative of the product of two functions, and the derivative of the quotient of two functions, are not simple like the above rules for addition and subtraction. You might expect that $(f(x) \cdot g(x))' = f'(x) \cdot g'(x)$ and $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)}{g'(x)}$, but that is not the case:

4. <u>Product Rule:</u> $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

In words, the derivative of the product of two functions is the derivative of the first times the second plus the first times the derivative of the second.

This rule can also be remembered by knowing what the rule would be if you're multiplying more than two functions. To find the derivative of $f(x) \cdot g(x) \cdot h(x)$, you would have $f(x) \cdot g(x) \cdot h(x)$ appear three times, with the prime going from f to g to h. That is,

$(f(x)\cdot g(x)\cdot h(x))' = f'(x)\cdot g(x)\cdot h(x) + f(x)\cdot g'(x)\cdot h(x) + f(x)\cdot g(x)\cdot h'(x).$

Example: Find $\frac{d}{dx}\, 5\,x^4 \sin x$.

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Here, we can consider the two functions being multiplied to be $5\,x^4$ and $\sin x$.

$$\frac{d}{dx}\, 5\,x^4 \sin x = \left(\frac{d}{dx}\, 5\,x^4\right)\cdot \sin x + 5\,x^4\cdot\left(\frac{d}{dx}\sin x\right)$$
$$= 20\,x^3 \sin x + 5\,x^4 \cos x\,.$$

5. <u>Quotient Rule:</u> $\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$.

(Some people remember this as lowdeehi minus hideelo over low-squared).

Example: Let $f(x) = \frac{x^2}{\tan x}$. Find $f'(x)$.

$$f'(x) = \frac{\tan x\cdot 2\,x - x^2\sec^2 x}{\tan^2 x} = \frac{2\,x\tan x - x^2\sec^2 x}{\tan^2 x}.$$

The rule that requires the most practice is the differentiation rule for the composition of functions, called the Chain Rule.

6. <u>Chain Rule:</u> $[f(g(x))]' = f'(g(x))\cdot g'(x)$.

The most common application of this rule occurs when you have a function raised to a power: $((f(x))^n)' = n(f(x))^{n-1}\cdot f'(x)$. For example, $\frac{d}{dx}\left(x^2+1\right)^{25} = 25\left(x^2+1\right)^{24}\cdot\frac{d}{dx}\left(x^2+1\right)$ $= 25\left(x^2+1\right)^{24}\cdot 2\,x = 50\,x\left(x^2+1\right)^{24}$.

Example: Find $\frac{d}{dx}\sin 3\,x^2$.

The first function to differentiate is the sine function (keeping the angle $3\,x^2$) and then multiply by the derivative of $3\,x^2$(which is $6\,x$). Be careful not to multiply the $6\,x$ with the angle $3\,x^2$: the $6\,x$ should be put in front of $\cos x$.

$$\frac{d}{dx}\sin 3\,x^2 = \cos 3\,x^2\,\frac{d}{dx}\,3\,x^2 = 6\,x\cos 3\,x^2.$$

For more complicated examples that involve the composition of three or more functions, you need to differentiate the outermost function first, and work your way to the inmost function.

Example: Find $\frac{d}{dx}\cos^2(\tan 5x)$. (Recall that $\cos^2 x = (\cos x)^2$).

First you have to consider the power, then the cosine function, then $\tan 5x$, for which you first differentiate the tangent function and then $5x$:

$$\begin{aligned}\frac{d}{dx}\cos^2(\tan 5x) &= 2\cos(\tan 5x)\cdot\frac{d}{dx}\cos(\tan 5x)\\ &= 2\cos(\tan 5x)\cdot(-\sin(\tan 5x))\cdot\frac{d}{dx}(\tan 5x)\\ &= -2\cos(\tan 5x)\cdot\sin(\tan 5x)\cdot\sec^2 5x\cdot\frac{d}{dx}5x\\ &= -10\cos(\tan 5x)\cdot\sin(\tan 5x)\cdot\sec^2 5x.\end{aligned}$$

After some practice, you should be able to do a problem like above all at one time, without carrying the $\frac{d}{dx}$ through the expression:

$$\begin{aligned}\frac{d}{dx}\cos^2(\tan 5x) &= 2\cos(\tan 5x)(-\sin(\tan 5x))\sec^2 5x\cdot 5\\ &= -10\cos(\tan 5x)(\sin(\tan 5x))\sec^2 5x.\end{aligned}$$

Example: Find the equation of the line tangent to the function $y = \sec^2 3x$ at $\left(\frac{\pi}{12}, 2\right)$.

$\frac{dy}{dx} = 2\sec 3x\cdot\sec 3x\tan 3x\cdot 3 = 6\sec^2 3x\tan 3x$. At $x = \frac{\pi}{12}$, $\frac{dy}{dx}$ is equal to 12.

Now apply the point-slope form of the equation of a line: $y - 2 = 12\left(x - \frac{\pi}{12}\right)$, or $y = 12x + (2 - \pi)$.

Example: A particle travels along a coordinate according to the equation $s(t) = 2t^3 - 3t^2 - 36t + 2$ where $s(t)$ is the position function (in meters) and t is in seconds ($t \geq 0$).

a) find the velocity $v(t)$.
b) find when the object is moving to the left and when it is moving to the right.
c) find the displacement of the particle in the first 4 seconds.
d) find the total distance traveled in the first 4 seconds.
e) find $a(3)$, where $a(t) = v'(t)$ is called the acceleration function.

a) $v(t) = 6t^2 - 6t - 36$.

b) First find where $v(t) = 0$: $6(t^2 - t - 6) = 0$ or $6(t-3)(t+2) = 0$, so $t = 3$.

Choosing, for instance, the number 1 from the interval $(0, 3)$, we find that $v(1) = -36 < 0$, so the particle is moving to the left for the interval $[0, 3)$, and similarly, since $v(4) = 36 > 0$, the particle is moving to the right on $(3, \infty)$.

c) The displacement is the final position minus the initial position, $s(4) - s(0) = -62 - 2 = -64$ meters.

d) Since the particle changes direction at $t = 3$, we should add the distances traveled from $t = 0$ to $t = 3$ and from $t = 3$ to $t = 4$, that is $|s(3) - s(0)| + |s(4) - s(3)| =$

$| -79 - 2 | + | -62 - (-79) | = 81 + 17 = 98$ meters.

e) $a(t) = 12\,t - 6$, so $a(3) = 30\ \frac{\text{meters}}{\text{sec}^2}$.

Implicit Differentiation

Sometimes, rather than y given as an expression in terms of x (such as $y = 3\,x^2 - 4\,x$), y is given implicitly in terms of x, for example in the equation $x^2\,y^2 - 3\,y^3 = 4$. This means that a value for x defines a value or values for y. A familiar example is the equation of the circle $x^2 + y^2 = 1$. For many such equations, it is often impossible rewrite the equation as y equal to an expression in x. Nevertheless, by the technique of implicit differentiation, it is possible to find the derivative of y. This is done by applying $\frac{d}{d\,x}$ to both sides of the equation. When $\frac{d}{d\,x}$ is applied to y, we simply write y' or $\frac{d\,y}{d\,x}$, and when finding the second derivative, we write y'' or $\frac{d^2\,y}{d\,x^2}$. (Be careful to distinguish between $\left(\frac{d\,y}{d\,x}\right)^2$, which means the square of the first derivative, and $\frac{d^2\,y}{d\,x^2}$). For these problems it is assumed that the equation determines y as a differentiable function of x.

Implicit differentiation is used in a type of word problem called related rates, where the variables are functions of time t, and equations are implicitly differentiated with respect to t.

Example: Find y' in terms of x and y for the equation $x^2\,y^2 - 3\,y^3 = 4$.

To differentiate, the product rule must be applied to $x^2\,y^2$. For $3\,y^3$, the power rule and the chain rule must be applied:

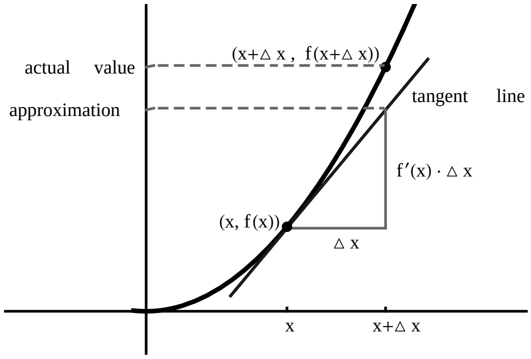
$$2\,x\,y^2 + x^2\,2\,y\,y' - 9\,y^2\,y' = 0$$
$$2\,x^2\,y\,y' - 9\,y^2\,y' = -2\,x\,y^2$$
$$y'(2\,x^2\,y - 9\,y^2) = -2\,x\,y^2$$

$$y' = \frac{-2\,x\,y^2}{2\,x^2\,y - 9\,y^2}\,.$$

Differentials and Linear Approximations

Expressions such as $d\,x$ or $d\,y$ are called differentials. (Differentials are different from the expressions $\frac{d}{d\,x}$ or D_x which ask you to find the derivative with respect to x). The definition, as a function on a vector space, is usually given in a course of Advanced Calculus. Here, differentials are used to provide what is called the linear approximation to a function at a point, and also error estimates.

One way to interpret the derivative of a function at a point is that it determines the slope of the line that best approximates the function near the point. If you know the value and the derivative of a function f at a point x, and are asked to approximate the value of f at a nearby point $x + \triangle x$ (where $\triangle x$ can be positive or negative), then the approximation is the y-coordinate of the point on the tangent line with x-coordinate $x + \triangle x$. The tangent line has slope $f'(x)$, so by the slope formula $\frac{\triangle\,y}{\triangle\,x} = f'(x)$, we see that the y-coordinate changes by an amount $\triangle\,y = f'(x) \cdot \triangle\,x$ from $f(x)$ (see the graph below).



To summarize:

$$f(x+\triangle x) \approx f(x) + f'(x)\cdot\triangle x.$$

Example: Estimate $(27.6)^{\frac{2}{3}}$.

The first step is to determine the function, and the point near 27.6 where the value of the func tion and its derivative can be evaluated without using a calculator. For this problem,

$$f(x) = x^{\frac{2}{3}} \qquad f(27) = 27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2 = 3^2 = 9$$

$$f'(x) = \frac{2}{3\,x^{\frac{1}{3}}} \qquad f'(27) = \frac{2}{3\cdot 27^{\frac{1}{3}}} = \frac{2}{9}, \text{ and } \Delta x = 27.6 - 27 = .6$$

$$f(27.6) \approx f(27) + f'(27)\cdot(.6) = 9 + \frac{2}{9}\cdot(.6) = 9 + \frac{2}{15} = \frac{137}{15}.$$

Example: Estimate $\cos 59.6\,°$.

The function to consider is $f(x) = \cos x$. However, degrees must be converted into radians. The angle near 59.6° where $\cos x$ and its derivative can be evaluated is 60° or $\frac{\pi}{3}$. Δx is $-.4\,°$ which is converted to radians by $(-.4)\cdot\frac{\pi}{180} = -\frac{\pi}{450} \approx -0.007$. Therefore:

$$\cos(59.6\,°) \approx \cos\left(\frac{\pi}{3}\right) + \left(-\sin\left(\frac{\pi}{3}\right)\cdot(-0.007)\right) = .5 + \left(\frac{\sqrt{3}}{2}\right)(0.007) \approx .5061.$$

Before proceeding, it can be reasonably asked the purpose of these linear approximations when a calculator can simply be used. Linear approximations are actually the first step in developing approximations to functions. If $f(x)$, $f'(x)$, and $f''(x)$ are known, then the best *quadratic* approximation near x can be found (that is, a function of the form $y = a\,x^2 + b\,x + c$) which provides a better approximation to the graph of f near x than the linear approximation. In the second semester of Calculus, for functions that are *infinitely differentiable* at a point, this process can be continued indefinitely to form expressions of the form $a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \,\ldots + a_n\,x^n + \,\ldots$ called power series that provide the *exact* value of the function nearby the point where the value of the functions and its derivatives are known. An approximation to the function to any degree of accuracy can be found by using a finite part of the power series.

Error Estimates

In this application, differentials are used to provide the best linear approximation to the change or error in one quantity given the change or error in a related quantity at specified values of the quantities. We'll use the notation of differentials to write the equation $\frac{dy}{dx} = f'(x)$ as $dy = f'(x) \cdot dx$.

Example: Suppose a container in the form of a cube is being constructed, and the volume it should contain is 1000 meters^3 to an accuracy of $\pm$ .03 meters^3. How accurate must the sides of the cube be made to guarantee this accuracy for the volume?

From the equation for the volume of a cube $V = s^3$, we find that the side should be

$$1000 = s^3 \text{ or } s = 10 \text{ meters.}$$

To find to what accuracy the side must be made, differentiate $V = s^3$:

$$\frac{dV}{ds} = 3s^2 \text{ and rewrite as } dV = 3s^2\, ds$$

dV is replaced with .03 and we solve for ds :

$$ds = \frac{.03}{3(10^2)} = \frac{.03}{300} = .0001.$$

The side must be made to within an accuracy of $\pm$.0001 meters to guarantee the volume is 1000 $\pm$ .03 meters^3.

Example: Waterproofing paint is to be applied to a spherical tank of radius 10 feet. If a coating of 0.05 inches is required, use differentials to estimate how many gallons of the waterproofing paint are needed (1 gallon $\approx$ 231 in^3).

Notice that the larger the container, the more paint has to be used. This problem asks you to find the amount of paint dV that is needed if a thickness of $dr = .05$ inches is applied to a sphere with radius = 10 feet = 120 inches.

Differentiate the volume formula for a sphere $V = \frac{4}{3}\pi r^3$ and substitute the above values:

$$dV = 4\pi r^2\, dr,$$
$$dV = 4\pi(120)^2\,(.05)$$
$$\approx 9047.79 \text{ in}^3.$$

To convert to gallons, divide this number by 231 to obtain about 39.2 gallons.