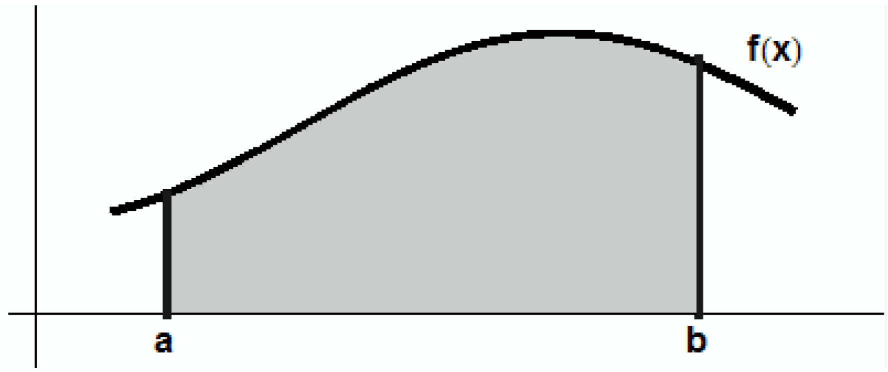
Integration

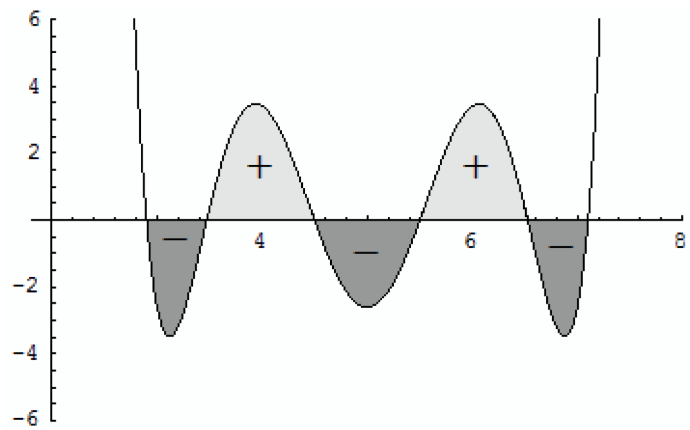
(Notes by Michael Samra)

The Definite Integral as Area

Let f be a continuous function on [a, b] , and for simplicity, suppose the graph of f lies above the the x-axis. Then the definite integral of f from a to b, written $\int_a^b f(x)\,dx$, represents the area between the curve and the x-axis from a to b (see below).

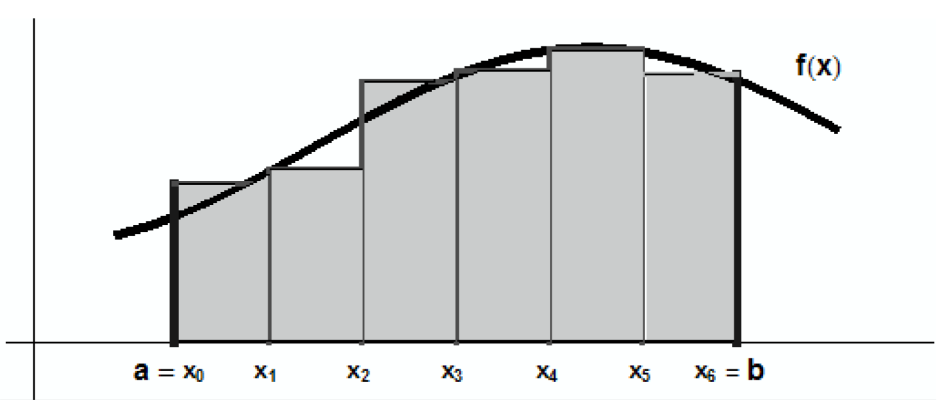


More generally, the definite integral of f from a to b will consist of positive pieces and negative pieces according to where f is above the x-axis and where it is below the x-axis. This is the signed area between f and the x-axis.



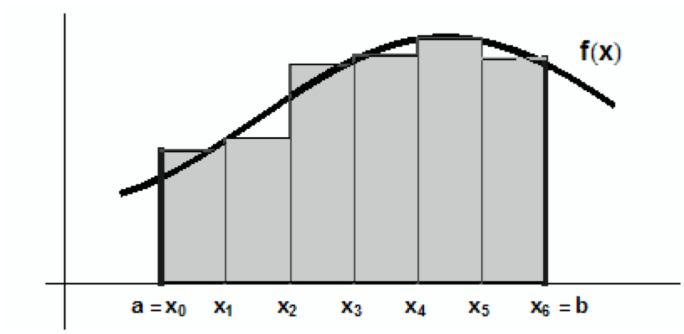
$\int_a^b f(x)\,dx$ gives a negative value for those pieces of f that lie below the x-axis

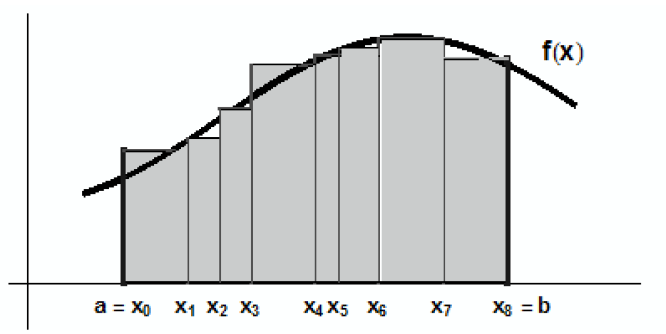
The definite integral is defined using a limit. The idea is to divide [a, b] into a number of smaller intervals called subintervals, and then to approximate the area between the curve and the x-axis by the sum of the areas of rectangles whose width is the length of the subinterval, and height the value of f at an arbitrarily chosen point in the subinterval. If a point x_i^* in the i-th subinterval is chosen where f is below the x-axis, $f(x_i^*)\cdot(x_i - x_{i-1})$ will contribute a negative number to the sum. A different definition makes use of what are called upper and lower sums, by using the maximum and minimum values of f in the subinterval.



The area under the curve is approximated by the area of rectangles. The i-th rectangle has width $\Delta x_i = x_i - x_{i-1}$, and height $f(x_i^*)$, where x_i^* is an arbitrarily chosen point in $[x_{i-1}, x_i]$. In this case, the subintervals have equal length Δx.

If the interval is further subdivided, you can see how this sum of rectangles can better approximate the area under the curve:





The limit is then taken as the length of the subintervals approach 0.

As is the case with differentiation, another way is found for calculating many integrals without having to use the definition. This is provided by The Fundamental Theorem of Calculus (FTC). The FTC says that calculating the integral of f on an interval [a, b] can be done by finding a function F, called an antiderivative, for which $F' = f$. $\int_a^b f(x)\,dx$ is then the difference of F at the endpoints of the interval, $F(b) - F(a)$.

Before presenting the FTC, the definition of the definite integral using Riemann sums, called the Riemann integral, is defined below. A very simple integral at the end of the section is calculated using the definition. After producing the same result using the FTC, one might wonder why it is worth bothering to calculate integrals using the definition. To answer this, integrals are used to calculate all sorts of quantities. Examples that are introduced in Calculus include volume, surface area, length of a curve or arc length, work, and fluid pressure. To do these problems, it is useful to determine the (infinitesimal) pieces that the integral adds up to produce the quantity. This amounts to finding the Riemann sum for the integral. Also, for integrals that do not have known antiderivatives, approximating the integral using rectangles is a first approach to approximating the integral. Better methods introduced in the second semester include the Trapezoidal Method and the Parabolic Method.

Riemann Sums

Before introducing the notation and definitions required to define Riemann sums, an example is given that should be helpful in understanding the definition of a Riemann sum and the idea of the definite integral as a limit of Riemann sums.

Example: Consider the continuous function $f(x) = x^2$ defined on $[1, 3]$.

a) Approximate the area under the graph of f on [1, 3] by dividing this interval into 4 equal subintervals, using the right-hand endpoints of the subintervals to evaluate the function (for the height of the rectangles).
b) Repeat a) using left-hand endpoints.
c) Repeat a) and b) using 8 equal subintervals.

a) The subintervals have size $\frac{3-1}{4} = 0.5$. The subintervals are [1, 1.5], [1.5, 2], [2,2.5], and [2.5, 3]. The right-hand endpoints are therefore 1.5, 2, 2.5, and 3. The approximation is the sum

$$.5 \cdot f(1.5) + .5 \cdot f(2) + .5 \cdot f(2.5) + .5 \cdot f(3) = .5\,(1.5)^2 + .5\,(2)^2 + .5\,(2.5)^2 + .5\,(3)^2$$
$$= .5\left(1.5^2 + 2^2 + 2.5^2 + 3^2\right) = 10.75.$$

b) The approximation is $.5\left(1^2 + 1.5^2 + 2^2 + 2.5^2\right) = 6.75$.

c) The subintervals have size $\frac{3-1}{8} = .25$. The approximation using the right-hand endpoints is

$$.25\left(1.25^2 + 1.5^2 + 1.75^2 + 2^2 + 2.25^2 + 2.5^2 + 2.75^2 + 3^2\right) = 9.6875.$$

Using left-hand endpoints: $.25\left(1 + 1.25^2 + 1.5^2 + 1.75^2 + 2^2 + 2.25^2 + 2.5^2 + 2.75^2\right) = 7.6875$.

Below, the actual value is calculated by using the limit of a Riemann sum, and later by using The Fundamental Theorem of Calculus. It is equal to $\frac{26}{3} = 8.6\overline{6}$. Though the approximations above are not very close, they did improve substantially when the 4 subintervals were each further subdivided. To obtain better approximations, more subintervals are needed, which produce longer sums of numbers. A compact way of representing sums of numbers is by using sigma (or summation) notation.

Sigma Notation

An expression of the form $\sum_{i=1}^{4} i^2$ ($\sum$ is the Greek capital letter sigma) is the sum of numbers obtained when i is replaced consecutively with 1, 2, 3, and 4. That is, $\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2$. The letter i, called the index of summation, itself is unimportant. $\sum_{k=1}^{4} k^2$, for example, represents the same sum.

Other examples:

$$\sum_{j=0}^{4} 2j = 0 + 2 + 4 + 6 + 8$$

$$\sum_{i=1}^{n} 1 = 1 + 1 + \ldots + 1 \text{ (n-times)} = n.$$

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n$$

$\sum_{i=1}^{n} a_i = a_1 + a_2 + \ldots + a_n$ (this is a general way of representing a sum of n numbers)

$\sum_{i=1}^{n} (a_{i+1} - a_i) = a_{n+1} - a_1$ (if you write out the terms of this sum, you will find that all numbers cancel in pairs except for the first and last; this is called a telescoping or collapsing sum)

$\sum_{i=1}^{n} \left((i+1)^3 - i^3\right) = (n+1)^3 - 1$ (this is a particular example of the formula above)

The following (linearity) properties hold for $\sum$:

a) $\sum_{i=1}^{n} c\, a_i = c \sum_{i=1}^{n} a_i$, c any constant.

b) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.

c) $\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$.

Note: it's not true that $\sum_{i=1}^{n} a_i \cdot b_i = \sum_{i=1}^{n} a_i \cdot \sum_{i=1}^{n} b_i$.

Useful formulas:

1) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ 2) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ 3) $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Proof: These can be proved by induction, or directly as follows. For 1), let $S = \sum_{i=1}^{n} i$. Then write S in two different ways and add:

$$\begin{array}{rl} S &= 1 + 2 + \ldots + (n-1) + n \\ + \ \underline{S} &\underline{= n + (n-1) + \ldots + 2 + 1} \\ 2S &= n(n+1) \end{array}$$

(when adding the two equations, there are n-many sums of $(n+1)$).

To prove 2), from the example above $\sum_{i=1}^{n} \left((i+1)^3 - i^3\right) = (n+1)^3 - 1 = n^3 + 3n^2 + 3n$. On the other hand, $\sum_{i=1}^{n} \left((i+1)^3 - i^3\right) = \sum_{i=1}^{n} \left(3i^2 + 3i + 1\right) = 3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$. Equating $3\sum_{i=1}^{n} i^2 + 3\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 = n^3 + 3n^2 + 3n$ and using the previous formulas, you can solve for $\sum_{i=1}^{n} i^2$. A similar proof can be done for 3).

We now proceed to define the definite integral using Riemann sums.

Definition: A partition P is a subset of $[a, b]$ that contains the numbers a and b. For convenience, P is usually written as $\{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \ldots < x_n = b$.

P divides $[a, b]$ into n subintervals, where the i-th subinterval has length $\Delta x_i = x_i - x_{i-1}$. The size (or norm) of the partition P is defined by $\|P\| = \max \Delta x_i$, that is, the width of the largest subinterval.

Definition: Let f be a function defined on a closed interval $[a, b]$, and P a partition of $[a, b]$ into n subintervals. For each $[x_{i-1}, x_i]$, let x_i^* be a sample point from the i-th subinterval. Then the expression

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$$

is called a Riemann sum for the partition P.

Refer to the graph on p.2 for an interpretation of this sum.

Definition: Let f be a function defined on a closed interval $[a, b]$. If $\lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*)\,\Delta x_i$ exists, then f is said to be integrable on $[a, b]$.

This limit is called the definite integral (or Riemann integral) of f from a to b and is written

$$\int_a^b f(x)\,dx.$$

a and b are called the limits of integration (not to be confused with the concept of limits), x is called the variable of integration, and the symbol $\int$ is meant to resemble an elongated Σ. The expression dx should not be confused with $\frac{d}{dx}$ or D_x. dx is called a differential, and its precise definition is given in a course in Advanced Calculus. It is useful notation in a technique of integration called u-substitution, and reminds one of Δx. dx is also referred to as an infinitesimal, and it's sometimes helpful to regard integrals as a sum of infinitesimal (arbitrarily small) elements.

The limit itself is quite complicated. If you are familiar with the definition of a limit, then to say that this limit is equal to some number L, means given $\epsilon > 0$, there exists $\delta > 0$ such that for all partitions P with $||P|| < \delta$ and any choice of sample points x_i^*, $|\sum_{i=n}^{n} f(x_i^*) - L| < \epsilon$.

It doesn't matter what letter is chosen as the variable of integration. So all $\int_a^b f(x)\,dx$, $\int_a^b f(t)\,dt$, $\int_a^b f(s)\,ds$ represent the same number.

If $[a, b]$ is partitioned into n equal subintervals of length $\Delta x = \frac{b-a}{n}$ then the Riemann sum has the simpler form

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \Delta x \sum_{i=1}^{n} f(x_i^*)$$

and the definite integral has the form

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x = \lim_{n\to\infty} \Delta x \sum_{i=1}^{n} f(x_i^*).$$

A reasonable question to ask is which functions are integrable. This is given by the following theorem:

<u>Theorem:</u> If f is continuous on [a, b], then it is integrable on [a, b].

This theorem depends on the fact that a continuous function on a closed interval is uniformly continuous on [a, b]. That is, given $\epsilon > 0$ there exists $\delta > 0$ such that for all c in [a, b], if $|x - c| < \delta$,
then $|f(x) - f(c)| < \epsilon$.

Example: Consider the function $f(x) = x^2$ defined on [1, 3].

a) Find the Riemann sum of f for the partition $\{1, 1.2, 1.9, 2.4, 3\}$ choosing right-hand endpoints as sample points.

b) Write the expression for the Riemann sum of f on [1, 3] with n equal subintervals using right-hand endpoints.

c) Find $\int_1^3 x^2\,dx$ using the above formulas.

a) $\Delta x_1 = 1.2 - 1 = .2$ $\quad f(1.2) = 1.44$
$\Delta x_2 = 1.9 - 1.2 = .7$ $\quad f(1.9) = 3.61$
$\Delta x_3 = 2.4 - 1.9 = .5$ $\quad f(2.4) = 5.76$
$\Delta x_4 = 3 - 2.4 = .6$ $\quad f(3) = 9$

$$R = (1.44)(.2) + (3.61)(.7) + (5.76)(.5) + (9)(.6) = 11.095.$$

b) We first refine the formula $\Delta x \sum_{i=1}^{n} f(x_i^*)$ given above.

The first right-hand endpoint x_1 at which the function is evaluated is $a + \Delta x$, and then Δx is added for each subsequent right-hand endpoint until $a + n \cdot \Delta x = b$ is reached. The x_i^* can therefore be replaced with $a + i \cdot \Delta x$, where i ranges from 1 to n.

Formula for the Riemann sum of f on [a, b] using n equal subintervals and right-hand endpoints as sample points:

$$\Delta x \sum_{i=1}^{n} f(a + i \cdot \Delta x).$$

For this problem, $\Delta x = \frac{3-1}{n} = \frac{2}{n}$, and so the Riemann sum is

$$\frac{2}{n} \sum_{i=1}^{n} f\left(1 + i \frac{2}{n}\right) = \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right).$$

c) $\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \left(1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right) = \lim_{n\to\infty} \frac{2}{n} \left(\sum_{i=1}^{n} 1 + \frac{4}{n} \sum_{i=1}^{n} i + \frac{4}{n^2} \sum_{i=1}^{n} i^2\right)$

$= \lim_{n\to\infty} \left(\frac{2}{n} \left(n + \frac{4}{n} \left(\frac{n(n+1)}{2}\right) + \frac{4}{n^2} \left(\frac{n(n+1)(2n+1)}{6}\right)\right)\right) = \lim_{n\to\infty} \left(2 + \frac{4n+4}{n} + \frac{8n^2+12n+4}{3n^2}\right)$

$= 2 + 4 + \frac{8}{3} = \frac{26}{3}.$

We will very shortly see how the integral that was found in c) can be done much more quickly without having to calculate a limit. This is provided by the fundamental result of Calculus that relates the operations of differentiation and integration.

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus, and what's sometimes called The Second Fundamental Theorem of Calculus, establish the sense in which differentiation and integration are actually inverse operations. The FTC provides a way of doing integration, by showing that evaluating the integral of f can be accomplished by finding a function F (called an antiderivative) for which $F' = f$. In this sense, integration can be understood as antidifferentiation.

Fundamental Theorem of Calculus:

Let f be continuous on [a, b]. If F is an antiderivative for f, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

The result of The Fundamental Theorem of Calculus is that differentiation formulas for functions can be translated into integration formulas. This gives the following integration formulas:

$\int x^r \, dx = \frac{x^{r+1}}{r+1} + C$, where r is a real number, $r \neq -1$.

$\int \sin x \, dx = -\cos x + C$ $\int \cos x \, dx = \sin x + C$

$\int \sec x \tan x \, dx = \sec x + C$ $\int \csc x \cot x \, dx = -\csc x + C$

$\int \sec^2 x \, dx = \tan x + C$ $\int \csc^2 x \, dx = -\cot x + C$

A special case of the first formula is $\int dx = \int x^0 \, dx = x + C$.

It is customary, for instance, to write all antiderivatives of cos x as sin x + C because any two antiderivatives differ by a constant. The formulas for the trigonometric functions can be remembered from the differentiation formulas. For example, since

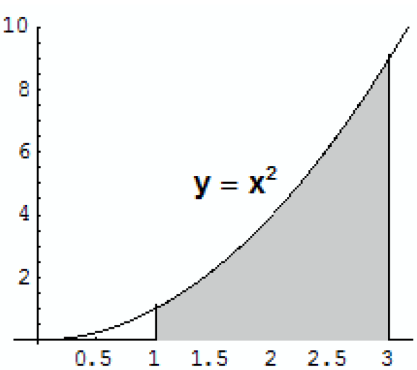
$\sec x \tan x = \frac{d}{dx} \sec x$,

$\int \sec x \tan x \, dx = \sec x + C$.

When evaluating definite integrals, the following notation is used: $F(x) \Big]_a^b = F(b) - F(a)$. a and b are called the limits of integration.

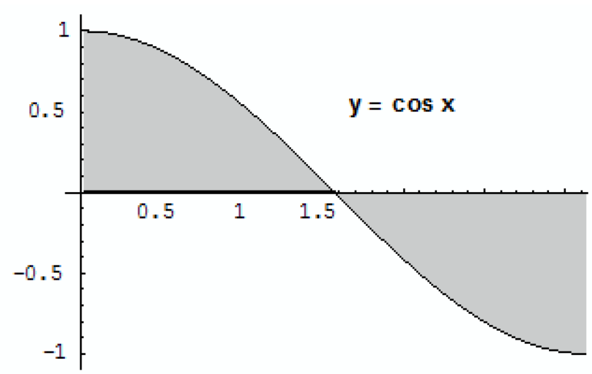
Example: Find $\int_1^3 x^2 \, dx$.

$$\int_1^3 x^2 dx = \frac{x^3}{3}\Big]_1^3 = \frac{3^3}{3} - \frac{1}{3} = \frac{26}{3}$$



Example: Find $\int_0^\pi \cos x \, dx$.

$$\int_0^\pi \cos x dx = \sin x\Big]_0^\pi = \sin\pi - \sin 0 = 0.$$



The two equal areas cancel each other out.

Properties of the Definite Integral

A consequence of the definition of the definite integral are the linearity properties of the integral. These properties can also be proved by using The Fundamental Theorem of Calculus.

1) $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$, c any real number.

2) $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

3) $\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$.

Note: it's not true that $\int_a^b f(x)\cdot g(x)\,dx = \int_a^b f(x)\,dx\cdot\int_a^b g(x)\,dx$.

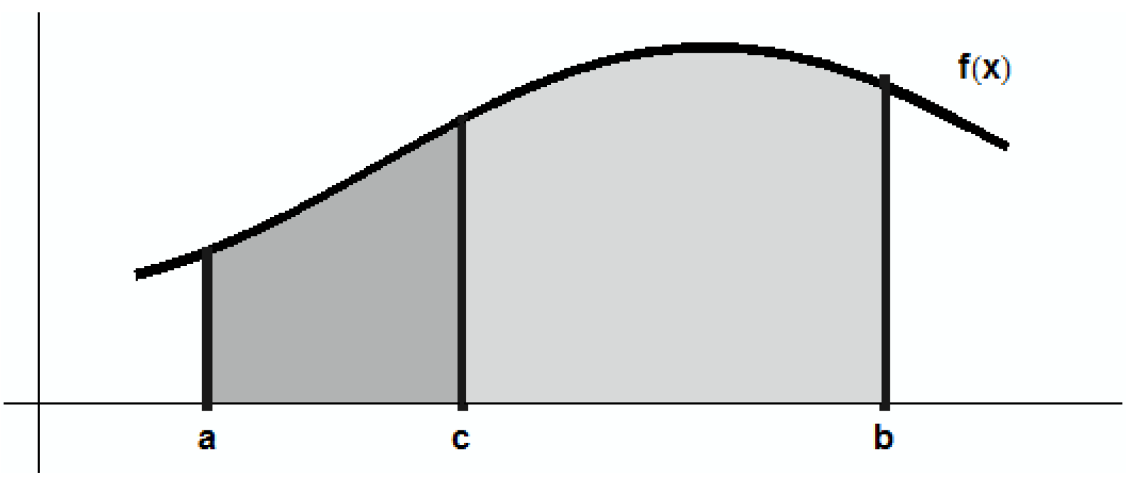
Three other properties of the definite integral are the following:

4) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ (this follows because $\frac{a-b}{n} = \frac{-(b-a)}{n}$).

5) $\int_a^a f(x)\,dx = 0$ (since $\Delta x_i = 0$ for each subinterval).

6) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, for any numbers a, b, and c (see below).

The case where $a < c < b$ is easily interpreted as saying that the area from a to b consists of the area from a to c plus the area from c to b (see the diagram below). The other cases are proved using Properties 4) and 5), or by using the FTC.



Example: Find $\int_0^\pi (3x + 2\sin x)\,dx$.

$$\int_0^\pi (3x+2\sin x)\,dx = \int_0^\pi 3x\,dx + \int_0^\pi 2\sin x\,dx = 3\int_0^\pi x\,dx + 2\int_0^\pi \sin x\,dx$$

$$= 3\,\frac{x^2}{2}\Big]_0^\pi - 2\cos x\Big]_0^\pi = \left(\frac{3\pi^2}{2} - 0\right) - (2\cos\pi - 2\cos 0) = \frac{3\pi^2}{2} - (-2-2) = \frac{3\pi^2}{2} + 4.$$

Example: Find $\int_1^2 \frac{5x^5 - x^2}{x^5}\,dx$.

$$\int_1^2 \frac{5x^5 - x^2}{x^5}\,dx = \int_1^2 \frac{5x^5}{x^5}\,dx - \int_1^2 \frac{x^2}{x^5}\,dx = \int_1^2 5\,dx - \int_1^2 x^{-3}\,dx$$

$$= 5x\Big]_1^2 + \frac{1}{2x^2}\Big]_1^2 = (10 - 5) + \left(\frac{1}{8} - \frac{1}{2}\right) = 5 + \left(-\frac{3}{8}\right) = \frac{37}{8}.$$

The Second Fundamental Theorem of Calculus

The Second Fundamental Theorem of Calculus completes the proof that differentiation and integration, in an appropriately defined way, are inverses to one another. The Second Fundamental Theorem of Calculus says that the derivative of the definite integral of a function with respect to the upper limit is the function at the upper limit. More precisely, the function $F(x) = \int_a^x f(t)\,dt$ is defined and it is shown that $\frac{d}{dx} F(x) = \frac{d}{dx} \int_a^x f(t)\,dt = f(x)$ for f a continuous function on $[a, b]$. Before proceeding, the notation will be explained, and then the function $F(x)$ will be described graphically.

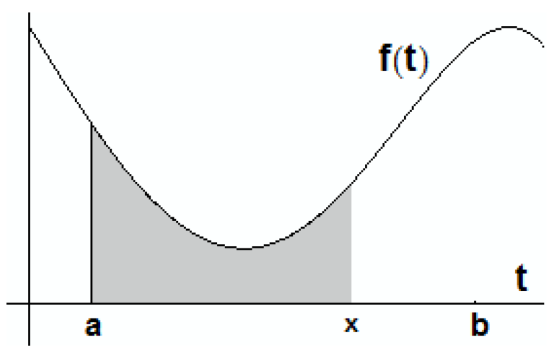
What often causes confusion when this function is first seen is that two variables, x and t, appear. Also, rather than the variable x being used as the variable of integration, t is being used. Though x is often used for the variable of integration, any of the integrals below are the same:

$$\int_a^b f(x)\,dx \qquad \int_a^b f(t)\,dt \qquad \int_a^b f(s)\,ds.$$

In other words, it doesn't matter what letter is used as the variable of integration. The limits of integration say in each case to find the (signed) area between the graph of f and the line $y = 0$ as the variable ranges from the numbers a to b. To define a function of x using the integral, it is necessary to use a different letter for the variable of integration.

To understand the function $F(x) = \int_a^x f(t)\,dt$, suppose for simplicity that f is positive on $[a, b]$.

Then for any choice of x in $[a, b]$, $F(x) = \int_a^x f(t)\,dt$ is the area under the curve of f between a and x (the shaded area in the diagram below).



The Second Fundamental Theorem of Calculus:

If f is continuous on $[a, b]$, the function F defined on $[a, b]$ by

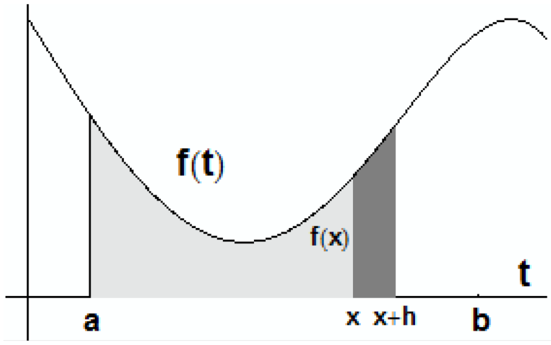
$$F(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$ and differentiable on (a, b), and has derivative

$$F'(x) = f(x) \text{ for all } x \text{ in } (a, b).$$

In Leibniz notation the conclusion of the theorem is written $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

The idea of the proof of this theorem can be demonstrated by referring to the diagram below:



$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \approx \frac{f(x)\cdot h}{h} = f(x)$. The numerator $F(x+h) - F(x)$ is the darker shaded area, which can be approximated by the rectangular area which has height $f(x)$ and width h.

Examples:

$$\frac{d}{dx}\int_1^x (t^2+1)^3\,dt = (x^2+1)^3. \qquad \frac{d}{dx}\int_4^x \cot 3t\,dt = \cot 3x.$$

If the upper limit of integration is a function of x, say g(x), then the chain rule has to be applied:

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x) = f(g(x))\cdot g'(x).$$

Example:

$$\frac{d}{dx}\int_2^{3x^2} (t^3+4)^5\,dt = \left((3x^2)^3+4\right)^5\cdot 6x = 6x(27x^6+4)^5.$$

Example:

$$\frac{d}{dx}\int_x^{\sin x}(t^2+1)\,dt = \frac{d}{dx}\left(\int_x^a (t^2+1)\,dt + \int_a^{\sin x}(t^2+1)\,dt\right) \quad \text{(by Property 6)}$$

$$= \frac{d}{dx}\left(-\int_a^x (t^2+1)\,dt + \int_a^{\sin x}(t^2+1)\,dt\right) \quad \text{(by Property 4)}$$

$$= -\frac{d}{dx}\int_a^x (t^2+1)\,dt + \frac{d}{dx}\int_a^{\sin x}(t^2+1)\,dt$$

$$= -(x^2+1) + (\sin^2 x + 1)\cos x,$$

where a is an arbitrary constant in the domain of the integrand.

Indefinite Integrals

Sometimes, instead of calculating the integral of a function over some interval, you are only asked to find the antiderivative of the integrand. In this case, the integral is written without the limits of integration, and is called an indefinite integral.

Example:

$$\int\left(x^{\frac{1}{4}} + \frac{\tan^2 x}{\sin^2 x}\right) dx = \int x^{\frac{1}{4}}\, dx + \int \sec^2 x\, dx = \frac{4}{5}\, x^{\frac{5}{4}} + \tan x + C.$$

Some types of problems require that the constant of integration C is found. One such type of problem occurs in physics. If an object traveling with velocity v(t) along a coordinate line is always traveling in the positive direction, then the integral of v(t) over the interval $[t_1, t_2]$ will be the distance the object traveled in that period of time. This might be expected, since velocity · time = distance. More generally, the integral of v(t) over the interval $[t_1, t_2]$ is the displacement of the object $s(t_2) - s(t_1)$, which is the difference between the end point and initial point of the object (where s(t) is the position function). To find s(t) given v(t), another piece of information has to be given, such as the initial position of theobject in order to find the constant C. The total distance traveled is given by the integral of |v(t)| from t_1 to t_2, or can be calculated according to where the velocity is 0, as is done below.

Example: An object travels along a coordinate line with velocity given by $v(t) = 2t - 4$ meters per second, with its position at time t = 1 two units to the right of the origin. Find the position function s(t). What is the distance it has traveled between t = 1 and t = 5?

Solution: $s(t) = \int v(t)\, dt = \int (2t - 4)\, dt = 2\int t\, dt - 4\int dt = t^2 - 4t + C.$

s(t) is specified by knowing the position at some moment of time, because then C can be found.

$$s(1) = 1^2 - 4(1) + C = 2$$
$$C = 5.$$

Hence $s(t) = t^2 - 4t + 5$. To find the total distance traveled, notice that the velocity is 0 when t = 2. This means that the object possibly changes direction at this time. The total distance traveled is therefore given by

$$|s(2) - s(1)| + |s(5) - s(2)| = |1 - 2| + |10 - 2| = 1 + 8 = 9 \text{ meters.}$$

Change of Variables (or u-substitution)

As mentioned, differentiation formulas for functions yield integration formulas. This also extends to to some of the differentiation rules that apply to operations involving functions. The rule for the differentiation of the composition of two functions (the Chain Rule) yields the technique of integration called change of variables or u-substitution. Next semester, the technique of integration corresponding to the Product Rule will be introduced, called integration by parts.

The technique of u-substitution is used to transform an integral that isn't part of the list of integrals that you know how to do, into an integral that is part of the list by changing the variable of integration. To derive this technique, start with the Chain Rule and then integrate both sides of the equation:

$$f'(g(x))\, g'(x) = \frac{d}{dx}(f(g(x))$$

$$\Longrightarrow \quad \int f'(g(x))\, g'(x)\, dx = \int \frac{d}{dx}\, f(g(x))\, dx$$

$$\Longrightarrow \quad \int f'(g(x))\, g'(x)\, dx = f(g(x)) + C$$

That is, given that you know an antiderivative for f' (which, of course, is f), then you can also find an antiderivative for $f'(g(x))\, g'(x)$, which is $f(g(x))$.

The technique of u-substitution is to first set $u = g(x)$. $\frac{d}{dx}$ is applied to both sides, but instead of writing $\frac{du}{dx} = g'(x)$, the du and dx are formally separated to form the equation $du = g'(x)\, dx$. The integral $\int f'(g(x))\, g'(x)\, dx$ is now written with the variable u. du replaces $g'(x)\, dx$ and the integrand $f'(u)$ is some function of u (u^4 in the example below) that hopefully can be integrated.

Example: Find $\int (x^3 + 4)^4\, x^2\, dx$.

$$\int (x^3 + 4)^4\, x^2\, dx$$

let $u = x^3 + 4$

$du = 3\, x^2\, dx$

The integral must be written entirely using the u variable: $(x^3 + 4)^4$ is replaced with u^4, and since $x^2\, dx$ must be replaced, use the equation $du = 3\, x^2\, dx$ to find that $\frac{1}{3}\, du = x^2\, dx$. The integral becomes:

$$\int u^4\, \frac{1}{3}\, du = \frac{1}{3} \int u^4\, du.$$

This is now an integral of a form that can be easily integrated:

$$\frac{1}{3} \int u^4\, du = \frac{1}{3}\, \frac{u^5}{5} + C$$

$$= \frac{(x^3+1)^5}{15} + C$$

The final answer is in terms of the original variable x.

This example follows the typical use of u-substitution. To anticipate whether a u-substitution will work, you can mentally find the derivative of u, and see whether it (or the expression multiplied by a constant) appears as part of the integrand. Also notice that you aren't setting u to be a function raised to a power, because the power is accounted for by writing u^4, which is easily integrated.

Example: Find $\int \cos 3\, x\, dx$.

$$\int \cos 3\, x\, dx$$

$\int \cos x\, dx$ can be done, which suggests proceeding as follows:

let $u = 3x$

$du = 3\,dx$

$(\frac{1}{3}\,du = dx)$

Then $\int \cos 3x\,dx = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin 3x + C.$

The following example is a little more difficult:

Example: Find $\int x\cos^3 x^2 \sin x^2\,dx.$

$\int x\cos^3 x^2 \sin x^2\,dx$

let $u = \cos x^2$

$du = -2x\sin x^2\,dx$

$(-\frac{1}{2}\,du = x\sin x^2\,dx)$ Then $\int x\cos^3 x^2 \sin x^2\,dx = -\frac{1}{2}\int u^3\,du = -\frac{1}{2}\frac{u^4}{4} + C$

$= -\frac{\cos^4 x^2}{8} + C.$

When u-substitution is used for evaluating definite integrals, it must be remembered that the limits of integration will change with the introduction of a new variable. For instance, if the original variable of integration x ranges from $\frac{\pi}{6}$ to $\frac{\pi}{2}$ and $u = \sin x$, then the new variable ranges from $\sin\frac{\pi}{6} = \frac{1}{2}$ to $\sin\frac{\pi}{2} = 1$. You have a choice to either evaluate the integral using u with the new limits of integration or replace u after doing the integral with the function of x it is equal to, and evaluate using the original limits of integration.

Both methods are illustrated below:

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int_1^9 u^{-\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{1}{2}}\Big]_1^9 = \frac{2}{3}\left(9^{\frac{1}{2}} - 1^{\frac{1}{2}}\right) = \frac{4}{3}.$$

let $u = x^3 + 1$ when $x = 0,\ u = 0^3 + 1 = 1$

$du = 3x^2\,dx$ when $x = 2,\ u = 2^3 + 1 = 9$

$(\frac{1}{3}\,du = x^2\,dx)$

$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx$

$u = x^3 + 1$

$du = 2x^2\,dx$

$(\frac{1}{3}\,du = x^2\,dx)$ Then $\int \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int u^{-\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{1}{2}} + C = \frac{2}{3}\sqrt{x^3+1} + C.$

$$\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{2}{3}\left(\sqrt{x^3+1}\right)\Big]_0^2 = \frac{2}{3}\left(\sqrt{9} - \sqrt{1}\right) = \frac{4}{3}.$$

If you write $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int_0^2 u^{-\frac{1}{2}}\,du = \frac{2}{3}u^{\frac{1}{2}}\Big]_0^2 = \frac{2}{3}\sqrt{x^3+1}\Big]_0^2 = \frac{2}{3}\left(\sqrt{9} - \sqrt{1}\right) = \frac{4}{3}$,

even though the correct answer appears at the end, it is incorrect to write $\frac{1}{3}\int_0^2 u^{-\frac{1}{2}}\,du$ because u does not range from 0 to 2 and it can be easy to forget to replace u with $x^3 + 1$ before evaluating.