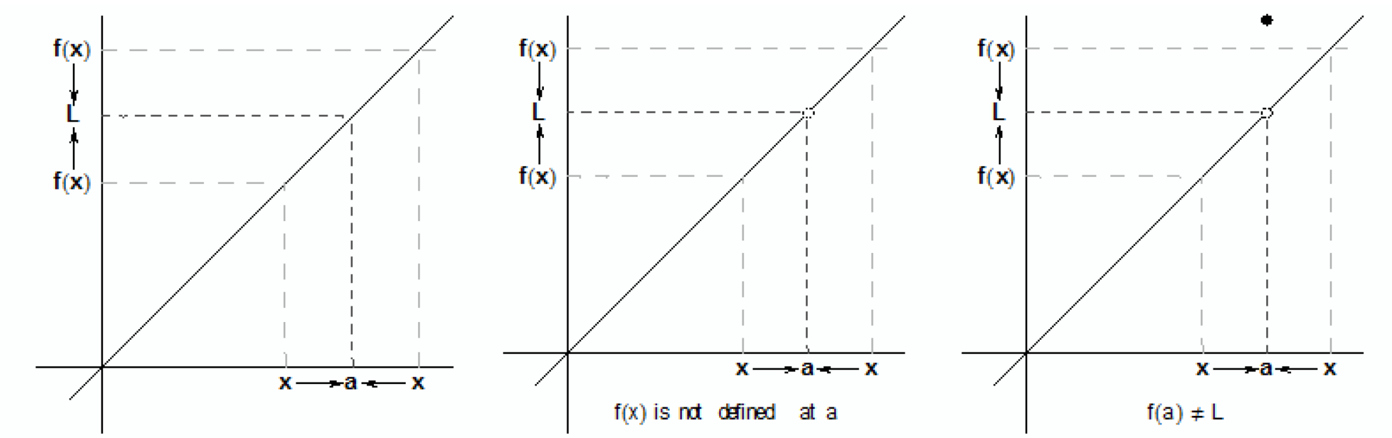
Limits and Continuity

(Notes by Michael Samra)

Limits are used throughout Calculus. They are used to define the continuity of a function, the derivative of a function, and the definite integral. In the second semester of Calculus, the limit of a sequence is defined which is used to define infinite sums, called infinite series. Usually, though, it is not until a course in Advanced Calculus that a student works closely with the technical definition of a limit.

The first use of a limit is to define the limit of a function at a point. Such a limit may, or may not exist. If it does exist, call this limit L. Then $\lim_{x \to a} f(x) = L$ means, in general terms, that as x approaches a (but not equal to a), $f(x)$ approaches L. Note that this definition does not depend on the actual value of the function at a: f may even not be defined at a, and we can still be asked to find $\lim_{x \to a} f(x)$.

In each of the 3 cases below, $\lim_{x \to a} f(x) = L$, even though for the second graph, f is not defined at a, and for the third graph, $f(a) \neq L$.

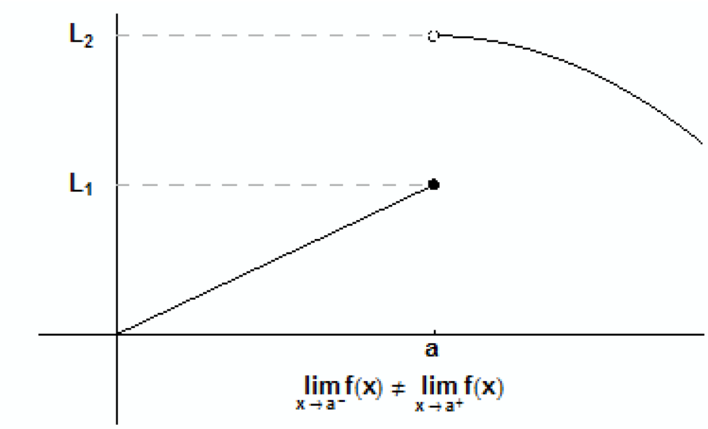


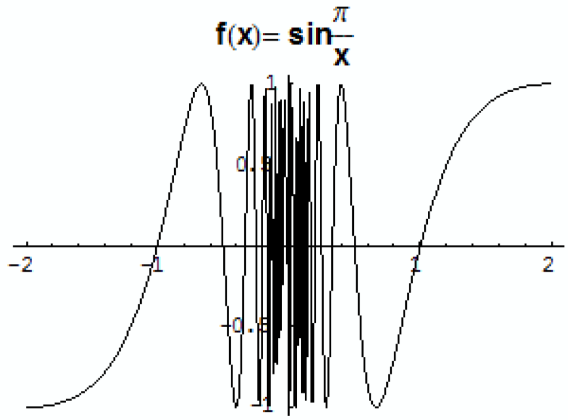
Since x can approach a number a from the left or from the right, the following theorem should make sense:

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

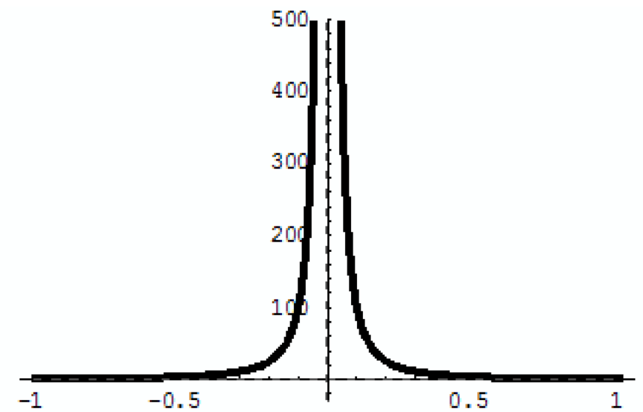
In other words, for the limit of a function f at a point a to exist, the limit as x approaches a from the left ($x \to a^-$) must equal to the limit as x approaches a from the right ($x \to a^+$).

How can the limit of a function fail to exist at a point? One way follows from the above theorem, and is illustrated by the first graph below, where $\lim_{x \to a^-} f(x) = L_1$ and $\lim_{x \to a^+} f(x) = L_2$ but $L_1 \neq L_2$. Another way is if the values of the function continue to oscillate as x approaches the number a. An example is given by $f(x) = \sin\frac{\pi}{x}$ as x approaches 0. $\sin\frac{\pi}{x}$ essentially takes all the oscillations of $\sin \pi x$ from $x = 1$ to ∞, and squeezes them in from $x = 1$ to $x = 0$: no matter how close x is to 0, the values of $f(x) = \sin\frac{\pi}{x}$ continue to oscillate between -1 and 1.





Example: Find $\lim_{x \to 0} \frac{1}{x^2}$.



From the graph one sees that $\frac{1}{x^2}$ does not approach any fixed number as $x \to 0$. Therefore, the limit does not exist. However, to indicate that $\frac{1}{x^2}$ increases without bound as $x \to 0$, one can write:

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Example: Find $\lim_{x \to 3^-} \frac{x+4}{x-3}$.

Here, the numerator is approaching 7 and the denominator is approaching 0, so the limit is either ∞ or $-\infty$. As x approaches 3 from the left, the numerator is positive and the denominator is negative, so the limit is $-\infty$.

The following rules for limits are used to evaluate limits:

Limit Rules

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

(i) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.

(ii) $\lim_{x\to a} c\, f(x) = c \lim_{x\to a} f(x)$, c any constant.

(iii) $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$.

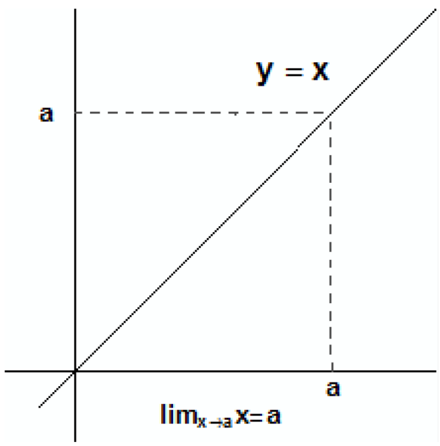
(iv) $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$.

(v) $\lim_{x\to a}(f(x))^n = [\lim_{x\to a} f(x)]^n$, n a positive integer.

(vi) $\lim_{x\to a}(f(x))^{\frac{1}{n}} = [\lim_{x\to a} f(x)]^{\frac{1}{n}}$, n a positive integer and $\lim_{x\to a} f(x) > 0$ for n even.

For many familiar functions, to find their limit at some number a, you simply have to evaluate the function at a. To show this for polynomial functions, first consider the function $f(x) = c\,x^n$. Applying the special limit

$\lim_{x\to a} x = a$ (see the graph below)



and limit rules (ii) and (v) yield: $\lim_{x\to a} f(x) = \lim_{x\to a} c\,x^n = c\lim_{x\to a} x^n = c\,a^n = f(a)$.

For polynomial functions $P(x) = c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0$, we can now apply limit rule (i) to obtain:

$$\lim_{x\to a} P(x) = \lim_{x\to a}(c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0) = c_n a^n + c_{n-1} a^{n-1} + \ldots + c_1 a + c_0 = P(a).$$

For quotients of polynomial functions $R(x) = \frac{P(x)}{Q(x)}$, called rational functions, apply limit rule (iv) to obtain:

$$\lim_{x\to a} R(x) = \lim_{x\to a}\frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} = R(a) \text{ (where } Q(a) \neq 0).$$

This is summarized in the following theorem:

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$\lim_{x\to a} P(x) = \lim_{x\to a} (c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0) = c_n a^n + c_{n-1} a^{n-1} + \ldots + c_1 a + c_0 = P(a)$.

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This is summarized in the following theorem:

Substitution Theorem:

Let f be a polynomial function or a rational function, and a in the domain of f. Then

$$\lim_{x\to a} f(x) = f(a).$$

Example: $\lim_{x\to 3} \frac{x^2+4}{2x-1} = \frac{3^2+4}{2\cdot 3-1} = \frac{13}{5}$.

The substitution theorem also holds for root functions and trigonometric functions.

Not all limits can be evaluated using the substitution theorem. A common example occurs when evaluation of the function at the number a yields an expression of the form $\frac{0}{0}$. For these problems, it is usually necessary to factor the numerator and/or denominator to eliminate common factors.

Example: Find $\lim_{x\to 2} \frac{x^2+2x-8}{x^2-4}$.

This function is not defined at x = 2. However, since $\frac{(x+4)(x-2)}{(x+2)(x-2)} = \frac{x+4}{x+2}$ for $x \neq 2$, (in other words, $\frac{x^2+2x-8}{x^2-4}$ has the same graph as $\frac{x+4}{x+2}$ except at x = 2, where it is not defined),

$$\lim_{x\to 2} \frac{x^2+2x-8}{x^2-4} = \lim_{x\to 2} \frac{x+4}{x+2} = \frac{3}{2}.$$

Example: Find $\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4}\right)$.

This limit has the form of $\infty - \infty$. To evaluate, combine the fractions, and then eliminate the common factor:

$$\lim_{x\to 2} \left(\frac{1}{x-2} - \frac{4}{x^2-4}\right) = \lim_{x\to 2} \frac{x+2-4}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x+2)(x-2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}.$$

Example: Find $\lim_{x\to 0} \frac{x}{\sqrt{3}-\sqrt{3-x}}$

Notice that the limit of both the numerator and denominator are 0. To find the limit, multiply the numerator and denominator by the conjugate of the denominator. This will eliminate the square roots in the denominator.

$$\lim_{x\to 0} \frac{x}{\sqrt{3}-\sqrt{3-x}} \cdot \frac{\sqrt{3}+\sqrt{3-x}}{\sqrt{3}+\sqrt{3-x}} = \lim_{x\to 0} \frac{x\left(\sqrt{3}+\sqrt{3-x}\right)}{3-(3-x)} = \lim_{x\to 0} \frac{x\left(\sqrt{3}+\sqrt{3-x}\right)}{x} =$$

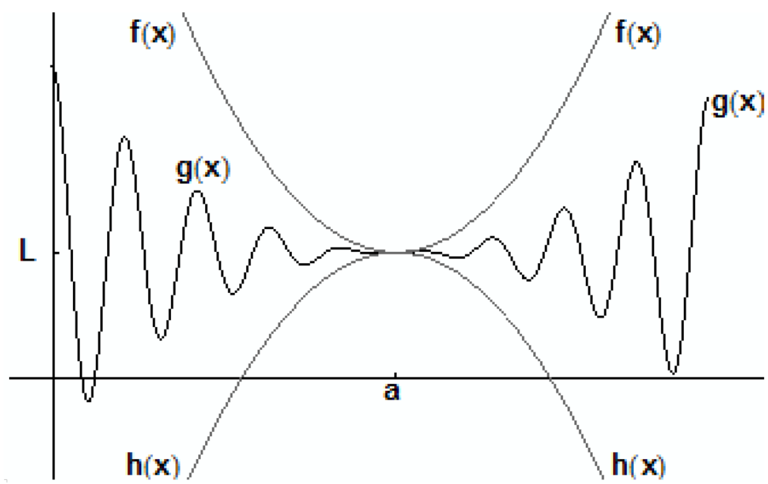
$$\lim_{x\to 0} \left(\sqrt{3} + \sqrt{3-x}\right) = 2\sqrt{3}.$$

A theorem that is useful in proofs involving limits is the following:

Squeeze Theorem (or Pinching Theorem) :

Suppose $f(x) \le g(x) \le h(x)$ for any x near a, and that $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then $\lim_{x\to a} g(x) = L$.

This theorem is illustrated by the graph below:



Example: Show $\lim_{x\to 0} x^2 \cos \frac{1}{\sqrt[3]{x}} = 0$.

Since $-1 \le \cos x \le 1$ (for any x), $-1 \le \cos \frac{1}{\sqrt[3]{x}} \le 1$. Hence, $x^2 \cdot (-1) \le x^2 \cdot \cos \frac{1}{\sqrt[3]{x}} \le x^2 \cdot 1$.

Furthermore, since $\lim_{x\to 0} (-x^2) = 0 = \lim_{x\to 0} x^2$, $\lim_{x\to 0} x^2 \cos \frac{1}{\sqrt[3]{x}} = 0$ by the Squeeze Theorem.

Continuity

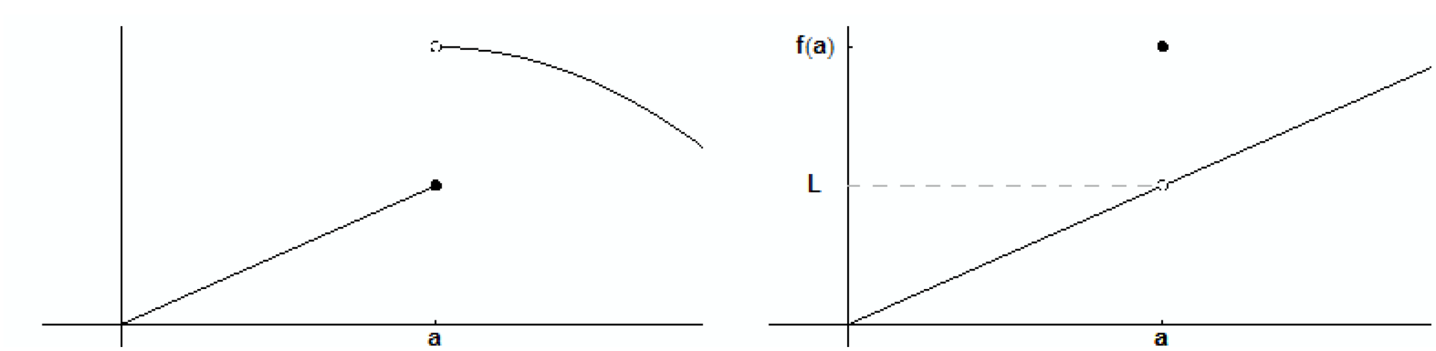
The methods of Calculus apply to functions that satisfy certain conditions. The first of these is continuity. In the next section, a more specific condition, called differentiability, is defined.

Definition: Let f be defined on an open interval containing a. Then f is continuous at a if:

(i) $\lim_{x \to a} f(x)$ exists
(ii) $\lim_{x \to a} f(x) = f(a)$

In other words, for a function to be continuous at a point a in its domain, the limit has to exist at a, and furthermore, that limit has be equal to the value of the function at a.

If condition (i) is violated, then f is said to have an essential singularity at a. If condition (i) holds, but condition (ii) is violated, then f is said to have a removable singularity at a (see the graphs below).



Example:

Let $f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ 2x, & x > 1 \end{cases}$.

Determine if $f(x)$ is continuous at $a = 1$.

Solution: First determine if the limit exists at 1. This is done by finding whether the limit from the left is equal to the limit from the right at 1:

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 1) = 2$. $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x) = 2$. Therefore $\lim_{x \to 1} f(x) = 2$.

Now find whether this number is equal to the value of f at 1:

$f(1) = 1^2 + 1 = 2$.

We conclude that since $\lim_{x \to 1} f(x) = 2 = f(1)$, f is continuous at $a = 1$.

Examples of functions that are continuous at every point in their domains include: polynomial functions, rational functions, root functions, and trigonometric functions. This follows from the Substitution Theorem above. Combinations of continuous functions are also continuous. More specifically:

Theorem: Suppose f and g are continuous at the point a. Then

i) $c\,f$, c any constant

ii) $f + g$

iii) $f - g$

iv) $f \cdot g$

v) $\frac{f}{g}$ (where $g(a) \neq 0$)

are continuous at a.

For example, the function $h(x) = \frac{\sin x}{x^2+3} + \sqrt[3]{x}$ is continuous at every real number.

Compositions of continuous functions are also continuous. In particular:

Theorem: Suppose g is continuous at a and f is continuous at g(a). Then $f \circ g$ is continuous at a.

For example, $h(x) = \sin^3\left(\sqrt{x-4}\right)$ is continuous wherever it is defined $(x \geq 4)$.

One reason for the importance of continuous functions is given by the following theorem:

Intermediate Value Theorem:

Suppose f is continuous on the closed interval $[a, b]$, and C is any number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) for which $f(c) = C$.

The Intermediate Value Theorem says that a continuous function on a closed interval $[a, b]$ doesn't skip any values between $f(a)$ and $f(b)$. It can be used to show that a solution to an equation exists.

Example: Use the Intermediate Value Theorem to show that $2x^3 + 5x - 9 = 0$ has a solution between 1 and 2.

Solution: Let $f(x) = 2x^3 + 5x - 9$. Since f is continuous, and $f(1) = -2$ and $f(2) = 17$, then for some c between 1 and 2, $f(c) = 0$.

Two methods that are often learnt in Calculus to approximate the location of c are the bisection method and Newton's method.