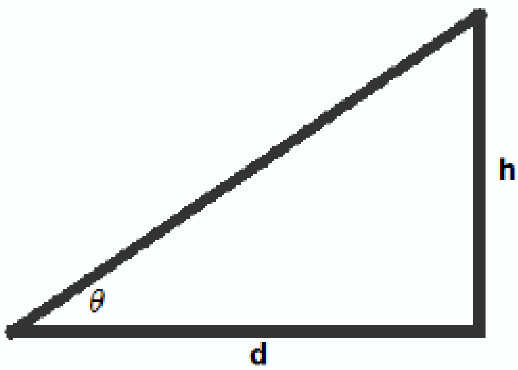
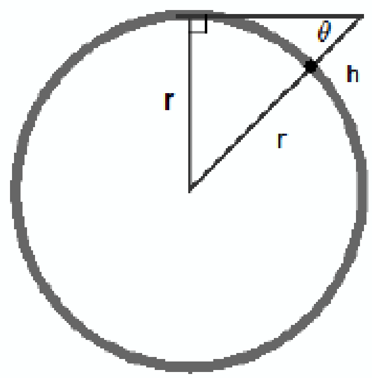
Trigonometry

(Notes by Michael Samra)

Introduction

Trigonometric functions were first developed to calculate distance and elevation in fields such as map surveying, navigation, and astronomy. Over 2000 years ago, the Greek mathematician and astronomer Hipparchus used the table of values of trigonometric functions he compiled to obtain amazingly accurate estimates for the radius of the earth and the distance to the moon.[1] First, the height of a mountain was found, and then the radius of the earth was found by measuring the angle formed by the line to the center of the earth and the line of sight to the horizon from the top of the mountain. See the diagrams below.

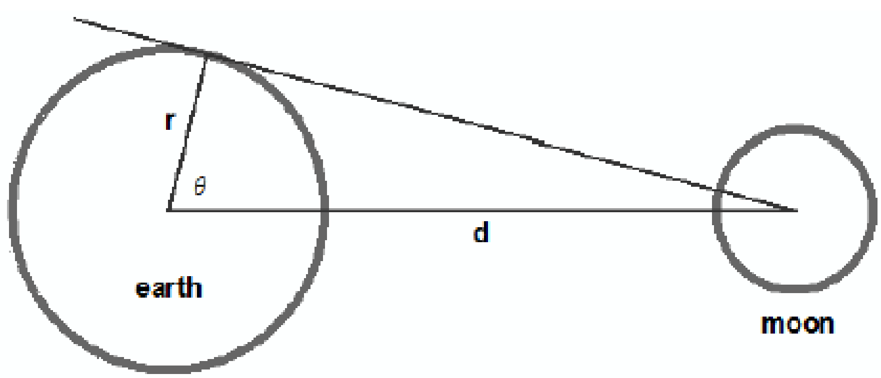




For simplicity, suppose the mountain is a vertical cliff of height h. Place the instrument to measure angles at a known distance d from the mountain and find θ. Then $\tan\theta = \frac{h}{d}$, or $h = d\tan\theta$.

Ascend a mountain of known height h and look toward the horizon to measure the angle θ. Then from $\tan\theta = \frac{r}{r+h}$, solve for r. The estimate Hipparchus obtained for r was 3944 miles. The actual value of the equatorial radius is 3953 miles.

He then used the radius of the earth to estimate the distance to the moon:

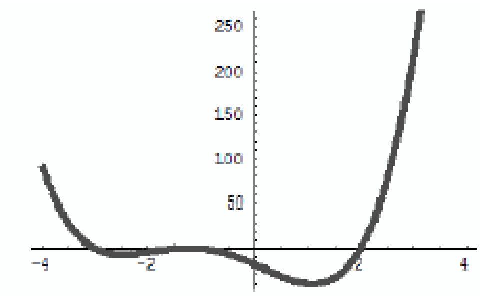


Suppose the estimate is made at a time when the equators of the earth and the moon are lined up. Then imagine a line drawn from the center of the moon tangent to the earth's surface. Measure the angle θ - it is the latitude of the point of tangency. Then $\cos\theta = \frac{r}{d}$, or $d = \frac{r}{\cos\theta}$. Hipparchus obtained the value of 238, 000 miles, where the actual mean value is 240, 290 miles.

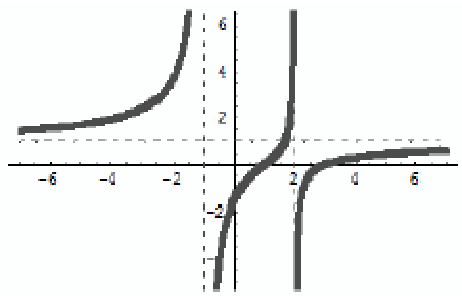
The second major application of trigonometry occurred with the development of calculus in the 17th century. At this time, trigonometric functions were defined as functions of real numbers using radian measure to make use of the methods of calculus. (See the last section of these notes for the definition of radian measure). The periodicity displayed by the graphs of the trigonometric functions were then applied to describe phenomena that exhibit periodic behavior such as the motion of a pendulum, planetary orbits, sound waves, and electromagnetic waves.

In a calculus course, however, trigonometric functions are studied for properties they share with other familiar functions such as polynomial functions and rational functions rather than for their special properties or applications.

$P(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$



$R(x) = \frac{x^2-4x+3}{x^2-x-2}$



Polynomial functions are functions of the form $P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$, where n is a nonnegative integer, and the a_i are real numbers.

Rational functions are of the form $\frac{P(x)}{Q(x)}$, where P(x) and Q(x) are polynomials.

In brief, students should know the values of the trigonometric functions at certain angles using radian measure instead of degrees ($\sin\frac{\pi}{6} = \frac{1}{2}$ rather than $\sin 30° = \frac{1}{2}$). The easiest way to do this is by using the unit circle, which is discussed below. In addition, students should be familiar with the graphs of the trigonometric functions and be able to solve trigonometric equations. Material not presented here that is used in the second semester are inverse trigonometric functions, and some of the trigonometric identities.

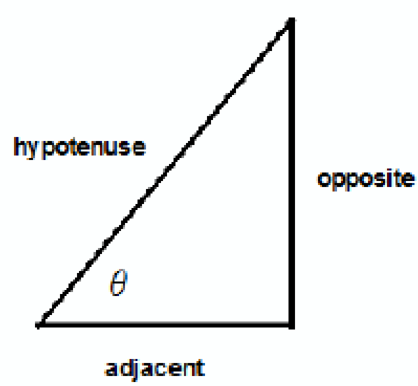
Definitions

The definition for right triangles :

Trigonometry is first learnt using right triangles. Given an acute angle θ in a right triangle, the definitions for the trigonometric functions appear below:

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$



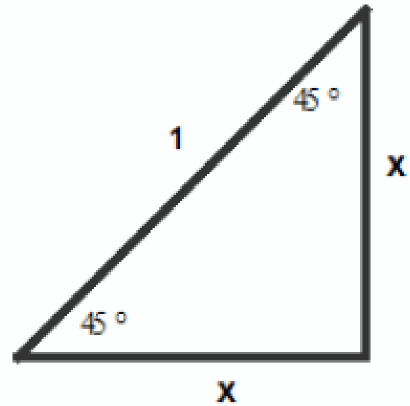
Two triangles for which we can easily find the values for the trigonometric functions are $45° - 45° - 90°$ and $30° - 60° - 90°$. For the first triangle, draw a triangle with hypotenuse of length 1, label the two equal sides x, and apply the Pythagorean Theorem:

$$x^2+x^2=1$$

$$2x^2=1$$

$$x=\sqrt{\frac{1}{2}}$$

$$x=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}$$

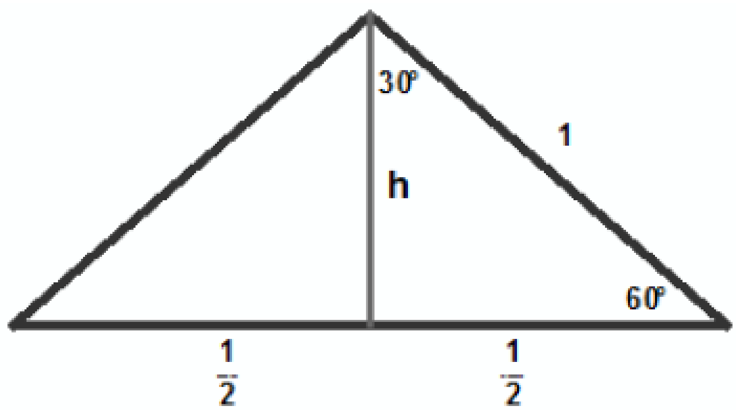


To obtain a $30° - 60° - 90°$ triangle, start with an equilateral triangle with sides of length 1, and draw the line from one of the vertices that bisects the opposite side. Then consider either of the two $30° - 60° - 90°$ triangles formed that have hypotenuse of length 1, and one of the legs of length $\frac{1}{2}$, and solve for the other leg that is labeled below as h:

$$h^2+\left(\frac{1}{2}\right)^2=1^2$$

$$h^2=\frac{3}{4}$$

$$h=\frac{\sqrt{3}}{2}$$



This provides us with the beginning of a table for the values of trigonometric functions at certain angles. The values for $\sec\theta$, $\csc\theta$, and $\cot\theta$ are found by computing the reciprocals of $\cos\theta$, $\sin\theta$, and $\tan\theta$.

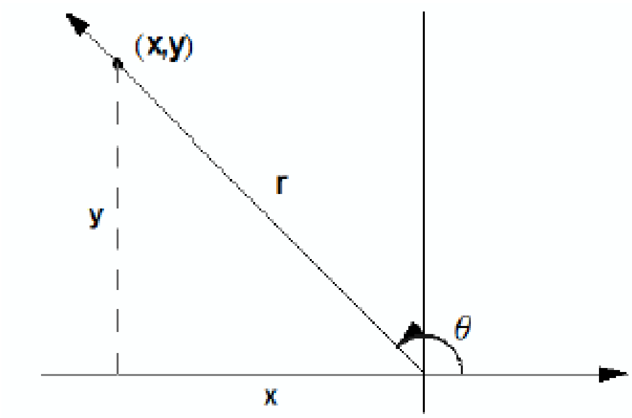
θ (degrees)	θ (radians)	$\cos\theta$	$\sin\theta$	$\tan\theta$
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$

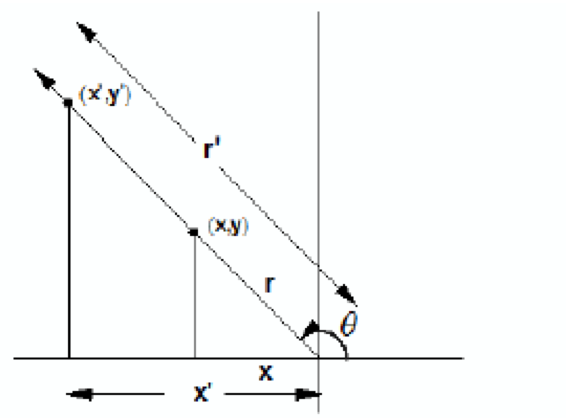
Degrees are converted to radians by multiplying by $\frac{\pi}{180}$, and to convert from radians to degrees, multiply by $\frac{180}{\pi}$. For example, $30°\cdot\frac{\pi}{180°}=\frac{\pi}{6}$.

<u>The definition for any angle θ</u>:

In precalculus, the definition is extended to any angle. An angle is said to be in standard position if its vertex is at the origin and the initial side is along the positive x-axis. Counterclockwise is considered the positive direction.

Let (x,y) be a point on the terminal side of an angle θ, and $r\left(=\sqrt{x^2+y^2}\right)$ the distance of (x,y) to the origin:





The definitions are as follows:

$\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$ $\tan\theta = \frac{y}{x}$

$\sec\theta = \frac{r}{x}$ $\csc\theta = \frac{r}{y}$ $\cot\theta = \frac{x}{y}$

(If another point (x', y') with distance r' from the origin is chosen along the ray, then the values of the trigonometric functions are the same by similar triangles; for $\cos\theta$, for example, $\frac{x'}{r'} = \frac{x}{r}$, as can be seen by the diagram above).

If you consider the acute angle made by the terminal side and the x-axis - in this case the negative part of the x-axis - the definitions are the same as for a right triangle.

Note that every trigonometric function can be defined using $\cos\theta$ and $\sin\theta$:

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

$\sec\theta = \frac{1}{\cos\theta}$ $\csc\theta = \frac{1}{\sin\theta}$

(Remember, for the bottom two, that rather than sine with secant and cosine with cosecant, it's just the opposite - s and c go together).

The Pythagorean Identities follow from the above definition:

$\sin^2\theta + \cos^2\theta = 1$ ($x^2 + y^2 = r^2$ implies $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$)

$\tan^2\theta + 1 = \sec^2\theta$ (The last two identities can be obtained from the first by dividing the entire equation by $\cos^2\theta$ or $\sin^2\theta$).

$1 + \cot^2\theta = \csc^2\theta$

Other trigonometric identities:

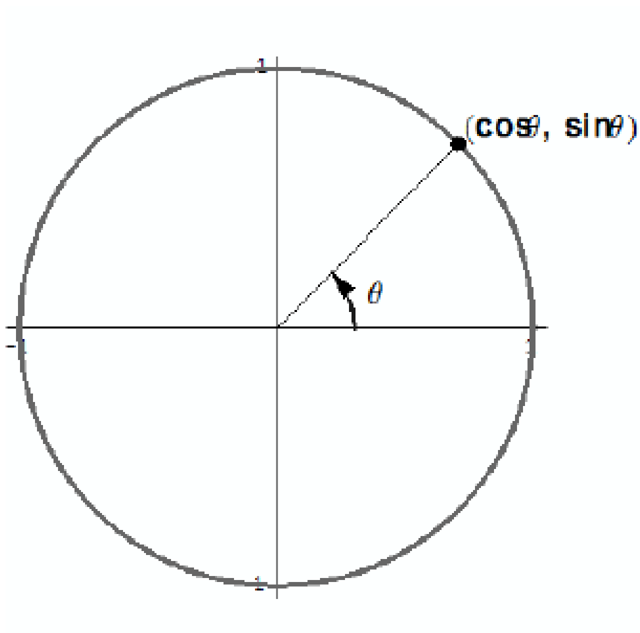
$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$

Notation:

Trigonometric functions are written differently than other functions you are familiar with that involve combinations of arithmetic expressions in x, such as $f(x) = 2x^2 + 2x - 1$. Here you can simply plug in the value for x to find y. The trigonometric functions, instead, are given names like sin x or cos x. One common mistake is to think, for instance, that cos 2 x is equal to 2 cos x. However, cos 2 x is the composition of $f(x) = \cos x$ with $g(x) = 2x$, that is $f(2x)$, which is different from $2 f(x) = 2\cos x$. For example, $1 = 2\cos\frac{\pi}{3} \neq \cos 2\cdot\frac{\pi}{3} = \cos\frac{2\pi}{3} = -\frac{1}{2}$.

The Unit Circle

The most useful way to remember values of the trigonometric functions is by using the unit circle. Since r = 1, the definitions for $\cos\theta$ and $\sin\theta$ reduce to $\cos\theta = x$ and $\sin\theta = y$. That is, any point (x , y) on the unit circle can be written as $(\cos\theta, \sin\theta)$. To emphasize this, $\cos\theta$ is the x-coordinate of the point, and $\sin\theta$ is the y-coordinate of the point on the unit circle (See the diagram below).



If the independent variable x is used instead of θ, then (cos x, sin x) represents the coordinates of the point on the unit circle. This can initially be confusing having a function cos x as the x-coordinate.

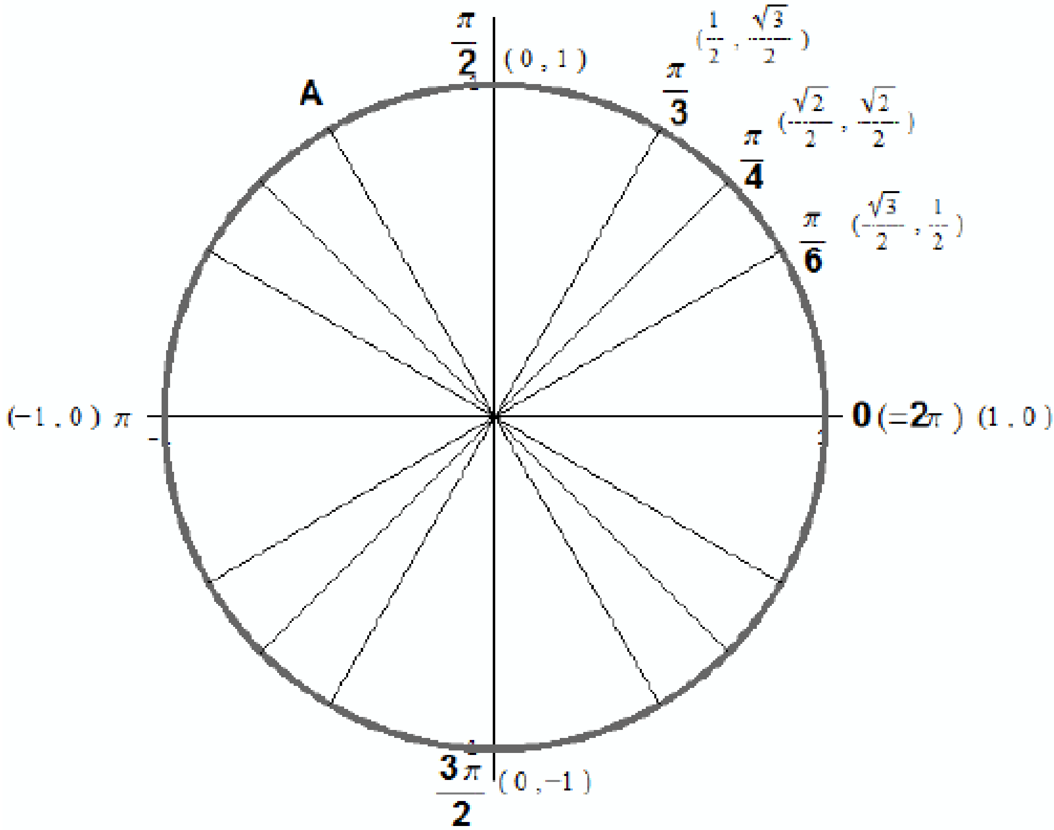
The definition using the unit circle:

Let (x,y) be a point on the unit circle on the terminal side of the angle θ. Then the definition of the trigonometric functions simplify to the following:

$\cos\theta = x$ $\qquad$ $\sin\theta = y$ $\qquad$ $\tan\theta = \frac{y}{x}$

$\sec\theta = \frac{1}{x}$ $\qquad$ $\csc\theta = \frac{1}{y}$ $\qquad$ $\cot\theta = \frac{x}{y}$

The unit circle can be used to display known values of the trigonometric functions. So far, we have the following:



For example, $\sec\frac{\pi}{3} = \frac{1}{\frac{1}{2}} = 2$. Also from the diagram we can find values of the trigonometric functions at integer multiples of $\frac{\pi}{2}$ (90 °). For example, $\sin\frac{3\pi}{2} = -1$.

The other angles that need to be filled in are integer multiples of $\frac{\pi}{6}$ (30 °), $\frac{\pi}{4}$ (45 °), and $\frac{\pi}{3}$ (60 °). These angles are called "reference angles" because the other angles will correspond to points on the unit circle that have the same coordinates in absolute value as one of these. The correct signs of the coordinates are determined according to which quadrant the point is in.

The reference angle of an angle θ is the difference (always taken to be positive) between the terminal side of θ and the horizontal axis. For angles in Quadrants II and III, this is the difference with the negative part of the x-axis, and for angles in Quadrant IV, it is the difference with the positive x-axis.

Example: Find the angle and coordinates on the unit circle that correspond to the terminal side labeled A.

The terminal side is in Quadrant II, so the reference angle is the difference with the negative part of the x-axis which you can see is $\frac{\pi}{3}$. The angle is then $\pi - \frac{\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3}$. The y-coordinate is the same as for $\frac{\pi}{3}$, but the x-coordinate is negative. The point on the unit circle is therefore $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

Important Exercise: Fill in the unit circle that appears on the last page.

Notice that if 2π is added to an angle θ or subtracted from an angle θ, the terminal side of the angle is the same. Therefore, the value of any of the trigonometric functions is unchanged if you add or subtract an integer multiple of 2π to the given angle θ.

Example: Find $\csc \frac{25\pi}{6}$

For positive angles, write the angle as the sum of an even multiple of π and an angle between 0 and 2π. Here 6 goes into 25 four times with remainder 1.

$$\csc \frac{25\pi}{6} = \csc\left(4\pi + \frac{\pi}{6}\right) = \csc \frac{\pi}{6} = \frac{1}{\frac{1}{2}} = 2.$$

Example: Find $\sin \frac{41\pi}{3}$

Here, 3 goes into 41 twelve times with remainder 5. (We can't use thirteen because it is not even).

$$\sin\left(\frac{41\pi}{3}\right) = \sin\left(12\pi + \frac{5\pi}{6}\right) = \sin \frac{5\pi}{6} = \frac{1}{2}.$$

Example: Find $\cos\left(-\frac{35\pi}{4}\right)$

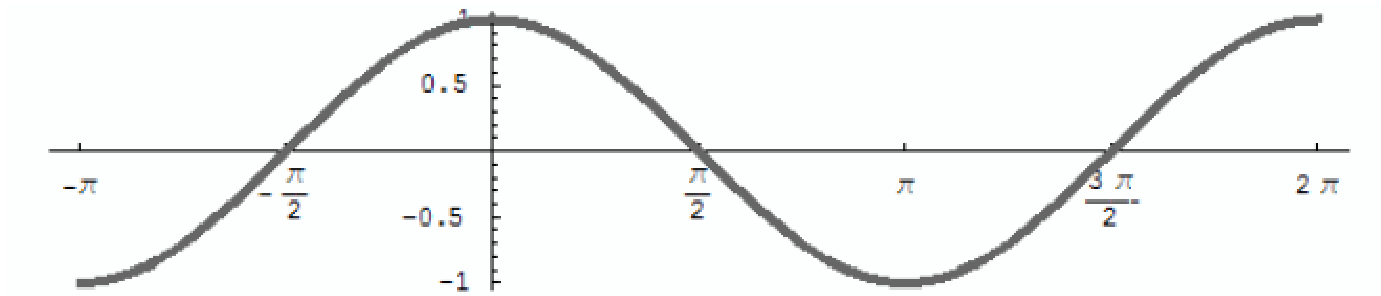
For negative angles, add an angle that is an even multiple of π and is larger in absolute value than the given angle. In this case, $\frac{40\pi}{4}$.

$$\cos\left(-\frac{35\pi}{4} + \frac{40\pi}{4}\right) = \cos\left(\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}.$$

Graphs of the Trigonometric Functions

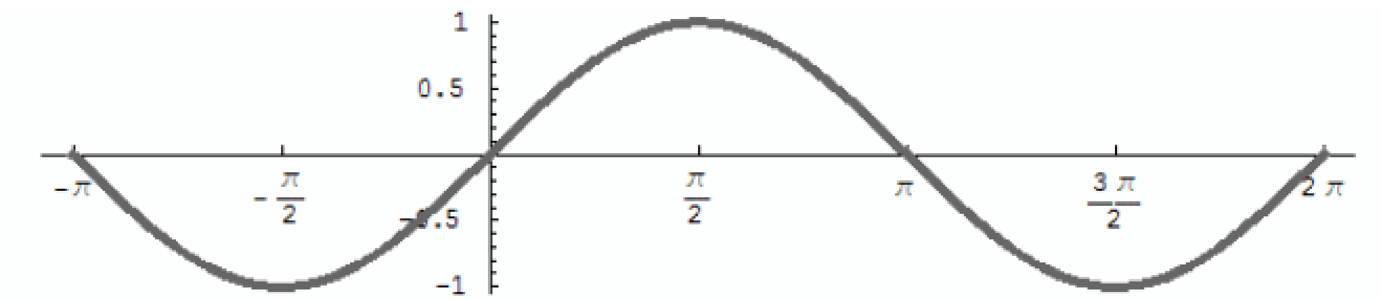
Below are the graphs of the 6 trigonometric functions. All have period 2π except for $y = \tan x$ and $y = \cot x$ which have period π. The graph of a function with period 2π repeats itself in intervals of 2π.

y = cosx



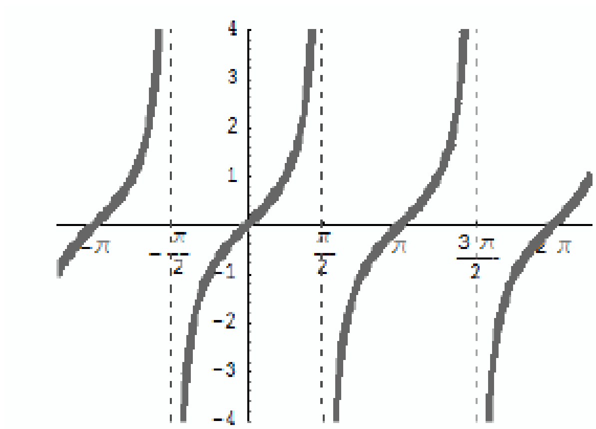
$y = \cos x$ is an example of an even function. Even functions satisfy $f(-x) = f(x)$. Graphically, even functions are symmetric with respect to the y-axis.

y = sin x

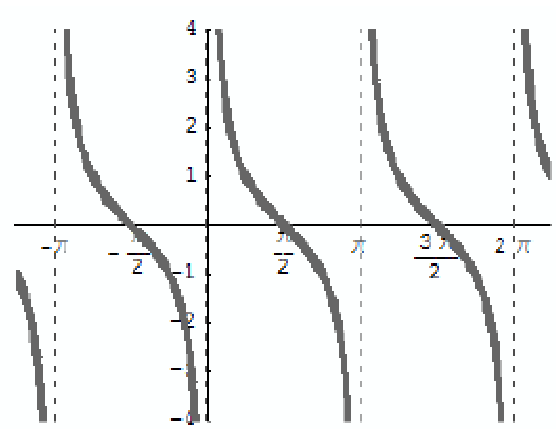


$y = \sin x$ is an example of an odd function. Odd functions satisfy $f(-x) = -f(x)$. Graphically, odd functions are symmetric with respect to the origin. The functions below, $y = \tan x$ and $y = \cot x$, are also odd functions.

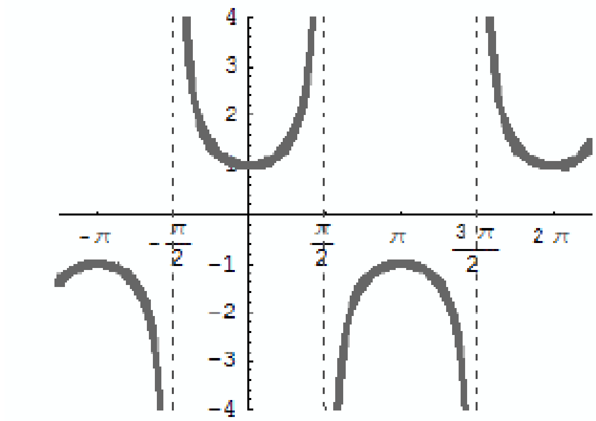
y = tan x



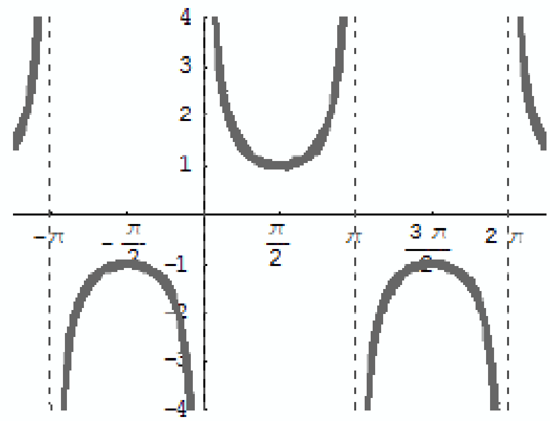
y = cot x



y = sec x



y = csc x



Trigonometric Equations:

The material in this section is used in Calculus I for graphing some different types of trigonometric functions. You might choose to wait to review this section until you encounter such graphing problems in your course.

Solving trigonometric equations is similar to solving ordinary equations, but with an additional step that involves knowing the angles for which trigonometric functions have certain values.

Example: Solve $2\sin x - 1 = 0$.

$$2\sin x = 1$$
$$\sin x = \frac{1}{2}$$

First, find the angles x in $[0, 2\pi)$ where $\sin x = \frac{1}{2}$:

$$x = \frac{\pi}{6} \quad \text{and} \quad x = \frac{5\pi}{6}$$

Next, since sin x has period 2π, adding or subtracting integer multiples of 2π to these angles are also solutions to these equations. Integer multiples of 2π can be represented by $2n\pi$, where n is any integer ($n = \ldots -2, -1, 0, 1, 2, \ldots$). So all solutions are given by:

$$x = \frac{\pi}{6} + 2n\pi,\ \frac{5\pi}{6} + 2n\pi, \quad n \text{ any integer.}$$

Example: a) Solve $2\sin 2x - 1 = 0$.
b) Solve $2\sin 2x - 1 = 0$ for x in $[0, 2\pi)$.

a) $2\sin 2x = 1$
$\sin 2x = \frac{1}{2}$.

Here, the angle $2x = \frac{\pi}{6} + 2n\pi$ or $2x = \frac{5\pi}{6} + 2n\pi$, and we solve for x by dividing these equations by 2:

$x = \frac{\pi}{12} + n\pi,\ \frac{5\pi}{12} + n\pi,$ n any integer.

b) Start with the solutions above and replace n with integers so that the angles obtained are in the interval $[0, 2\pi)$:

$x = \frac{\pi}{12}, \frac{13\pi}{12}$ or $x = \frac{5\pi}{12}, \frac{17\pi}{12}$ (where $n = 0, 1$)

Altogether, $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$.

Example: Solve $2\cos^2 x + 3\cos x = -1$. (Recall $\cos^2 x = (\cos x)^2$).

This equation is similar to a quadratic equation of the form $2x^2 + 3x = -1$. To solve, bring the -1 to the other side, and factor:

$$2\cos^2 x + 3\cos x + 1 = 0$$
$$(2\cos x + 1)(\cos x + 1) = 0$$

$2\cos x + 1 = 0$ or $\cos x + 1 = 0$

$2\cos x = -1$

$\cos x = -\frac{1}{2}$ $\qquad$ $\cos x = -1$

$x = \frac{2\pi}{3} + 2n\pi,$ $\qquad$ $x = \pi + 2n\pi.$

$\frac{4\pi}{3} + 2n\pi.$

Altogether: $x = \frac{2\pi}{3} + 2n\pi,\ \pi + 2n\pi,\ \frac{4\pi}{3} + 2n\pi,$ n any integer.

Example: Find all solutions of $\tan^2 3x = \tan 3x$ in the interval $[0, \pi)$.

$$\tan^2 3x - \tan 3x = 0$$
$$\tan 3x(\tan 3x - 1) = 0$$

$\tan 3x = 0$ or $\tan 3x = 1$

$3x = 0 + n\pi$ $\qquad$ $3x = \frac{\pi}{4} + n\pi$ (since $\tan x$ has period π, we need only look for solutions in $[0, \pi)$ and add $n\pi$).

$x = \frac{n\pi}{3}$ or $x = \frac{\pi}{12} + \frac{n\pi}{3}$ (dividing both sides by 3)

$x = \frac{n\pi}{3}$ yields the solutions $0, \frac{\pi}{3}, \frac{2\pi}{3}$. $x = \frac{\pi}{12} + \frac{n\pi}{3}$ yields the solutions $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$.

Altogether: $x = 0, \frac{\pi}{12}, \frac{\pi}{3}, \frac{5\pi}{12}, \frac{2\pi}{3}, \frac{3\pi}{4}$.

Radian Measure versus Degree Measure

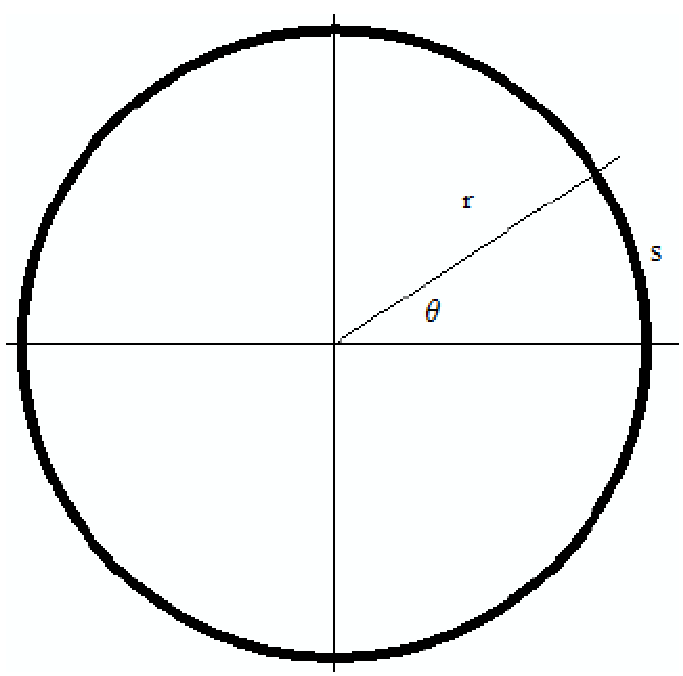
In Calculus, radians is mostly used instead of degrees. This is why it's important to remember angles using radians, though it is more time consuming having to add fractions (to find $\frac{3\pi}{4} + \frac{2\pi}{3}$, adding the fractions $\frac{3}{4} + \frac{2}{3}$) than to add whole numbers ($135° + 120°$).

There are several reasons for this. Algebraically, if x represents an angle in degrees, then expressions like $\sin x + x$ and $\frac{\sin x}{x}$ do not make sense. How do you add a real number (since $\sin x$ is a real number between -1 and 1) and a degree, or divide a real number by a degree? Also, graphing presents a problem, because how do you decide the length along the x-axis that is supposed to measure $1°$?

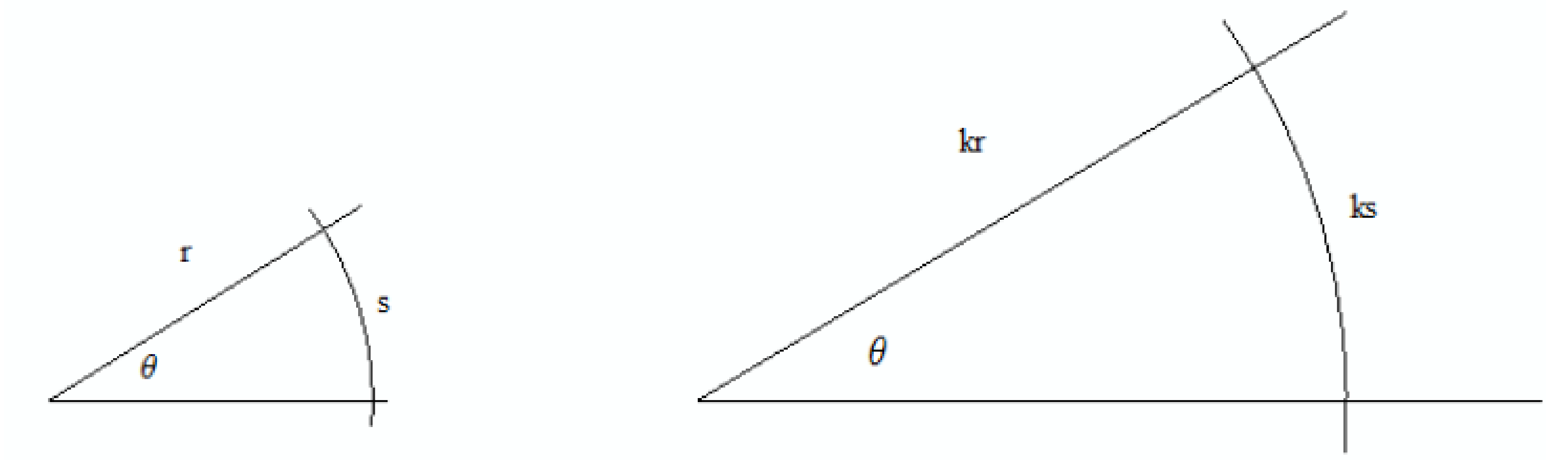
The functions that are studied in the first two semesters of Calculus are those with domain the real numbers (or a subset of the real numbers), and range the real numbers (or a subset of the real numbers). This is necessary to be able to perform the operations that are learnt in Calculus. For instance, knowing the limit of $\frac{\sin x}{x}$ as x approaches 0 is necessary in finding the derivatives of the trigonometric functions.

The definition of radian measure is as follows: let s be the length of the arc of a circle of radius r of a sector with angle θ (see the picture below). Then the radian measure of θ is defined to be $\theta = \frac{s}{r}$. This relationship is often introduced as $\theta = s\ r$.

$$\theta = \frac{s}{r}$$



First notice that $\frac{s}{r}$ is a real number, since both s and r have the units of length, and so the quotient is dimensionless (i.e. does not have units). Secondly, the definition should not depend on the radius of the circle. Observe that any two sectors with central angle θ are similar (just like two triangles with the same angles are similar). So if the radius of the new sector is k r, then the length of the arc is k s, and
$\frac{k\,s}{k\,r} = \frac{s}{r}$.



[1] Morris Kline, Mathematics in Western Culture, Oxford University Press, 1978.

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