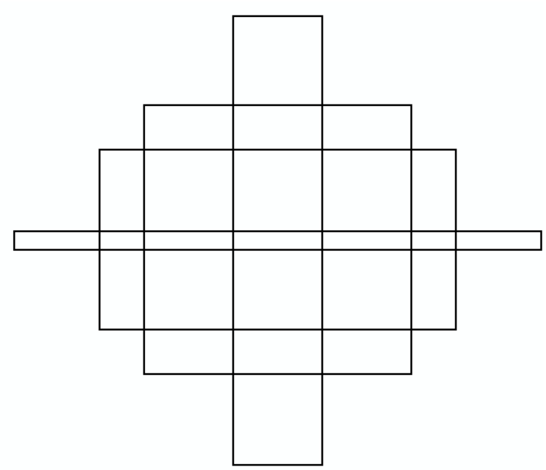
Word Problems

(Notes by Michael Samra)

Optimization word problems ask you to maximize or minimize some quantity or function given some relationship or constraint between the variables that are involved. For example: 24 yards are to be used to enclose a rectangular garden. Find the dimensions of the rectangular garden with maximum area that can be enclosed with the given fencing. In other words, amongst all rectangles of perimeter 24 yards (this gives the relationship between the variables length and width), find the dimensions of the rectangle of largest area.



Each rectangle has the same perimeter of 24, but not the same area.

Actually, this fencing problem is often done in Precalculus. The area function to be maximized is a quadratic function ($A = l\,w = w(12 - w) = -w^2 + 12\,w$), and for quadratic functions the maximum (or minimum) occurs at the vertex. To find the maximum or minimum of more complicated functions, themethods of Calculus need to be employed.

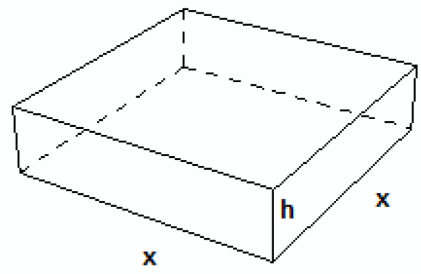
Optimization Examples

Optimization problems (also called maximum-minimum problems) occur in many fields and contexts in which it is necessary to find the maximum or minimum of a function to solve a problem. In economics, for example, companies want to find the level of production that maximizes profit. In physics, Fermat's Law of Optics states that light travels along a path that requires the least time. If light has to travel from a point A in the atmosphere to a point B in the ocean, an optimization problem is to find the path that minimizes the time. A practical application that's done in the second example is the following: a cable television company has to run cable across a river to a town that is some miles downstream. Cable is more expensive underwater than on land. Given the prices of cable and the distances, what is the least expensive way to lay the cable?

Three examples of optimization problems are presented, along with the steps to use to approach these problems.

Example 1: An open box with square base is to be constructed from 108 square inches of material. What dimensions will produce the greatest volume?

Underlined "Step 1:" *Draw a diagram or picture, if possible, labeling appropriately with variables (and possibly constants).*



Step 2: *Write an equation for the quantity that must be maximized or minimized.*

$$V = x^2 h$$

Step 3: *Write the quantity to be maximized or minimized as a function of one variable. Use the information in the question, or methods such as similar triangles or the Pythagorean Theorem, to find a relationship between the variables.*

In this example, we're told that 108 square inches of material is used. That is, the surface area is 108 square inches. The bottom has area x^2 and there are four sides each with area $x \cdot h$. This gives the equation:

$$x^2 + 4\,x\,h = 108$$

Now, solve for h and substitute into the equation above to write the volume as a function of one variable:

$$h = \frac{108 - x^2}{4\,x}, \quad \text{so} \quad V(x) = x^2\left(\frac{108 - x^2}{4\,x}\right)$$

$$V(x) = -\frac{x^3}{4} + 27\,x$$

Step 4: *Find the (appropriate) domain of your function.*

As a function, $V(x) = -\frac{x^3}{4} + 27\,x$ is defined for all real numbers x. However, for this problem, we know that we must have $x > 0$, and $x < \sqrt{108}$ (because for $x = \sqrt{108}$, the bottom has area 108 square inches, which leaves no material for the sides). Therefore, the appropriate domain is $\left(0, \sqrt{108}\right)$.

The reason that you should state the domain of the function is that if you have a closed interval or half-closed interval as the domain, the absolute maximum (or minimum) can occur at one of the endpoints of the domain.

Step 5: *Find the critical points of the function, and use either the first derivative test or second derivative test to show that the critical point is the maximum (or minimum). If the function is defined on a closed interval, you must also check the endpoints of the domain to determine where the absolute maximum (or minimum) is achieved.*

Differentiating $V(x)$ we find:

$$V'(x) = -\frac{3x^2}{4} + 27$$

Set this equal to zero and solve:

$$-\frac{3x^2}{4} + 27 = 0$$
$$\frac{3x^2}{4} = 27$$
$$x^2 = 36$$
$$x = \pm 6$$

To show that the maximum volume is achieved when $x = 6$, we'll use the second derivative test:

$$V''(x) = -\frac{3x}{2}$$
$$V''(6) = -9 < 0$$

Since the second derivative is negative when $x = 6$, the volume will be maximized at this value for x.

Note: sometimes you can show that the critical point is the desired maximum (or minimum) without having to apply either of the derivative tests. For this example, if you consider $V(x)$ as a function defined on the closed interval $\left[0, \sqrt{108}\right]$, then $V > 0$ except at the endpoints where $V = 0$, so the critical point must be a maximum.

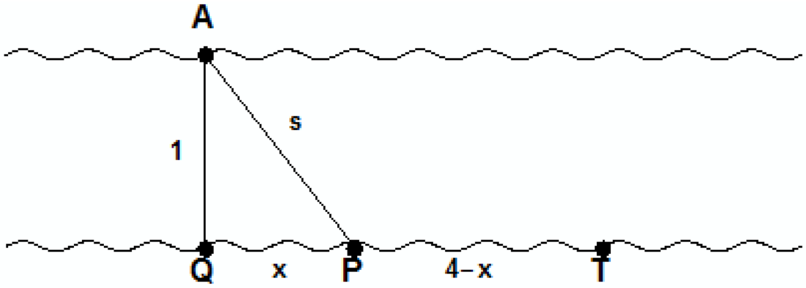
Step 6: *Answer the question, and check that your solution makes sense.*

We were asked to find the dimensions of the box, so we need to find what h is when $x = 6$:

$$h = \frac{108-6^2}{4 \cdot 6} = 3.$$

The dimensions of the box are 6"x 6" x 3".

Example 2: A cable television company has its main antenna located at point A on the bank of a straight river 1 kilometer wide. It is going to run a cable from point A to point P on the opposite side of the river and then straight along the bank to a town T situated 4 kilometers downstream from A. It costs \$15,000 per kilometer for cable underwater, and \$9000 per kilometer along the bank. What should the distance from P to T be to minimize cost?



Note: the distance from P to T could be labeled as x and the distance from Q to P as $4 - x$; the labeling above makes the calculations a little easier.

We want to minimize the cost C of the cable: $C = 15,000\,s + 9000\,(4 - x)$. To write this as a function of one variable, use the Pythagorean Theorem to solve for s: $s = (1 + x^2)^{\frac{1}{2}}$. Hence, the function to minimize is:

$$C(x) = 15,000\,(1 + x^2)^{\frac{1}{2}} + 9000\,(4 - x)$$

Differentiating:
$$C'(x) = 15,000 \cdot \frac{1}{2}\,(1 + x^2)^{-\frac{1}{2}}\,2\,x - 9000$$
$$= 15,000\,x(1 + x^2)^{-\frac{1}{2}} - 9000$$

Find critical point(s):
$$15,000\,x(1 + x^2)^{-\frac{1}{2}} - 9000 = 0$$
$$5\,x(1 + x^2)^{-\frac{1}{2}} = 3$$
$$5\,x = 3\,(1 + x^2)^{\frac{1}{2}}$$

Squaring both sides:
$$25\,x^2 = 9\,(1 + x^2)$$
$$16\,x^2 = 9$$
$$x = \pm\frac{3}{4}$$

To check that $x = \frac{3}{4}$ yields a minimum, find $C''(x)$:

$$C''(x) = 15,000\,(1 + x^2)^{-\frac{1}{2}} + 15,000\,x\left(-\frac{1}{2}\right)(1 + x^2)^{-\frac{3}{2}} \cdot 2\,x$$
$$= 15,000\,(1 + x^2)^{-\frac{1}{2}} - 15,000\,x^2(1 + x^2)^{-\frac{3}{2}}$$

Factoring out $15,000\,(1 + x^2)^{-\frac{3}{2}}$ yields:

$$C''(x) = 15,000\,(1 + x^2)^{-\frac{3}{2}}[(1 + x^2) - x^2] = \frac{15,000}{(1+x^2)^{\frac{3}{2}}}$$

Since $C''\left(\frac{3}{4}\right) > 0$, the minimum cost occurs at $x = \frac{3}{4}$. To answer the question, the distance from P to T should be $4 - \frac{3}{4} = 3\ \ \frac{1}{4}$ kilometers.

Divide the string into two pieces, x and $28-x$, and use the first piece for the circle and the second piece for the square. In other words, the circumference of the circle is x and the perimeter of the square $28-x$, and we need to calculate the corresponding areas.

For the circle: we need to find the radius to express the area. Since $2\pi r = x$, $r = \frac{x}{2\pi}$ and so the area is given by $\pi\left(\frac{x}{2\pi}\right)^2 = \frac{x^2}{4\pi}$.

For the square: we need to find the length of the side of the square. Since the perimeter is $28-x$, the length of a side is given by $\frac{28-x}{4}$, and so the area is given by $\left(\frac{28-x}{4}\right)^2$.

Altogether: $A(x)=\frac{x^2}{4\pi} + \left(\frac{28-x}{4}\right)^2$. The appropriate domain is [0, 28].

Differentiating: $A'(x) = \frac{x}{2\pi} - \frac{2(28-x)}{4}$

$$A'(x) = \frac{x}{2\pi} + \frac{x}{2} - 14$$

$$A'(x) = x\left(\frac{1+\pi}{2\pi}\right) - 14$$

Setting $A'(x) = 0$, you find $x = \frac{28\pi}{1+\pi}$.

Finding $A''(x)$: $A''(x) = \frac{1+\pi}{2\pi} > 0$ for any choice of x. Therefore, $x = \frac{1+\pi}{2\pi}$ yields a minimum.

The maximum must therefore occur at one of the endpoints: when $x = 0$, the area is 48 square inches; when $x = 28$, the area is $\frac{196}{\pi}$ square inches. Since $\frac{196}{\pi} > 48$, the maximum occurs when the entire string is used for the circle.

Summary

Related Rates

Finding the rate of change of some quantity given information about other, related quantities

***Step 1*:** draw a diagram, if possible, labeling with variables those quantities that vary in time and with numbers those quantities that remain constant.

***Step 2*:** using mathematical notation, write what's given and what you're asked to find.

Optimization (or Max-Min)

Maximizing or minimizing some quantity or function

***Step 1*:** draw a diagram, if possible, labeling appropriately with variables (and possibly constants).

***Step 2*:** write an equation for the quantity that must be maximized (or minimized).

***Step 3*:** write an equation involving the quantities in the problem.

***Step 4*:** implicitly differentiate the equation with respect to time.

***Step 5*:** plug in the given quantities to solve.

***Step 3*:** write the quantity to be maximized (or minimized) as a function of one variable.

***Step 4*:** find the domain of the function appropriate to the problem.

***Step 5*:** find the critical points of the function, and use the first or second derivative test to show the chosen critical point is the desired extremum (or use the domain, as explained on p. 7). If the domain is a closed interval, endpoints must also be checked.

***Step 6*:** answer the question posed in the problem, and check that the solution is sensible.