10. Conservation of Linear Momentum

Introduction:

The last module showed there is a different way of looking at Newton's Laws of Motion involving conservation of energy. But energy is not the only conserved quantity in physics and another conserved quantity is linear momentum. There actually two kinds of momentum: one associated linear motion and one associated with angular motion (called angular momentum). Angular momentum obeys a separate conservation law and this will be discussed in a later module.

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Definition of Linear Momentum:

Suppose a mass M has velocity V, then we say it has Kinetic Energy KE = $\frac{1}{2}$ M V^2. Additionally we say the mass has linear momentum $\vec{P}=M\vec{V}$ and shortly we will see why this definition is important.

1. First notice that linear momentum $\vec{P}$ is a vector quantity since it is proportional to the velocity $\vec{V}$. (This is in contrast with Kinetic Energy which is a scalar quantity.)

2. Consider the momenta $\vec{P}$ of two objects having different masses M > m but the same velocity $\vec{V}$:

The object with the greater mass M has more momentum. So momentum is different from velocity.

3. Units of Momentum: kg×m/sec

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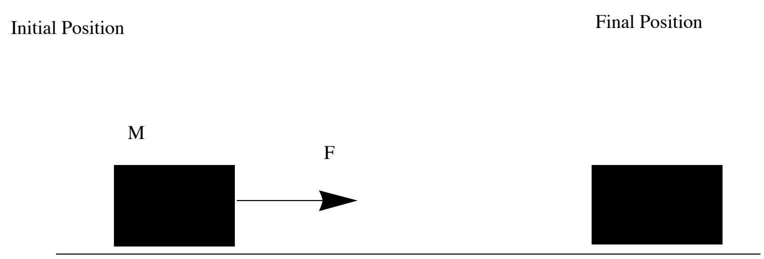
Introduction of Momentum into Physics

Suppose a force $\vec{F}$ acts for a time Δt on a mass M as pictured below.



Combine

1. Newton's 2nd Law: $\vec{F} = m\,\vec{a}$ plus

2. Definition of acceleration: $\vec{a} = \frac{\vec{V}_f - \vec{V}_o}{\Delta t}$

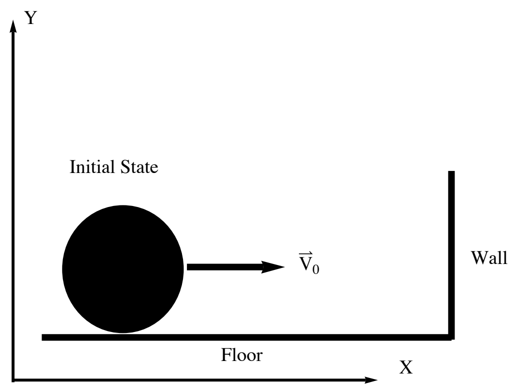
to yield $\vec{F} = m\,(\frac{\vec{V}_f - \vec{V}_o}{\Delta t})$ which after a slight rearrangement is $\vec{F}\,\Delta t = m\,\vec{V}_f - m\,\vec{V}_o$

Using $\vec{P}_f = M\vec{V}_f$ for the definition of linear momentum after the force has acted and using $\vec{P}_o = M\vec{V}_o$ for the momentum before the force has acted allows us to write

$\vec{F}\,\Delta t = \vec{P}_f - \vec{P}_o$

Interpretation: Impulse = $\vec{F}\,\Delta t$ causes a change in momentum $\Delta\vec{p} = \vec{P}_f - \vec{P}_o$

Example:
Calculation of the Change in Momentum of a Ball Hitting a Wall



The X, Y coordinate system is as pictured.

$V_o > 0$ is the initial velocity is positive.

$V_f < 0$ because it is in the negative X direction.

Assume $V_f = -V_0$ which is the same thing as assuming energy is conserved: $\frac{1}{2} M V_0^2 = \frac{1}{2} M V_f^2$

1. The initial momentum originally before the ball hits the wall is $P_o = MV_o$

2. The final momentum after the ball bounces off of the wall is $P_f = M V_f = M(-V_0)$.

3. The change in momentum $\Delta p = P_f - P_o = M(-V_0) - MV_o = -2 MV_o$ (which is NOT zero).

NUMERICAL EXAMPLE: Suppose M=5 kg, $V_o = 3$ m/s and $V_f = -V_o = -3$ m/s. The change in momentum ΔP of the ball is

$\Delta P = -2 \times 5 \text{ kg} \times 3 \text{ m/s} = -30 \text{ kg} \times \text{m/sec}$

$F\Delta t = \Delta P = -30 \text{ kg} \times \text{m/sec}$ (the minus sign means the force is to the left away from the wall.)

Suppose $\Delta t = 0.2$ sec is the time the ball is in contact with the wall.

The average force F of the wall on the ball $F = \frac{\Delta P}{\Delta t}$

$F = (-30 \text{ kg} \times m/\text{sec})/(0.2 \text{ sec}) = -150 \text{ Nt}$

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30 / .2
150.
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Rocket Propulsion: How do they work?

Burning fuel (hydrogen, kerosene etc.) leaves the rocket engine with high velocity since it is very hot and since the gas molecules have mass, momentum is leaving the rocket motor. The rest of the rocket payload moves in the opposite direction. This is the basic principle behind the rocket.

Typical Example: 1300 kg/s of rocket fuel leaves the motor with a velocity of 50, 000 m/s.

Calculate the thrust (which is the force in Newtons). Conservation of Linear Momentum applied to the rocket yields

$F\,\Delta t = m\,V_f - m\,V_0$ or $F = \frac{m}{\Delta t} V_f - \frac{m}{\Delta t} V_0$

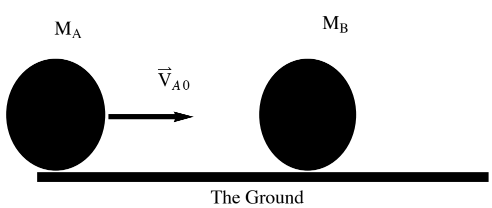
where m is the amount of fuel burned in a time Δt, V_f = 50,000 m/s, and $V_f \simeq 0$ m/s (since the unburned fuel is moving with the rocket and the rocket speed is small (even if the rocket is moving at the velocity of sound) compared with 50,000 m/s the speed of the burnt gas. The rate at which the fuel is burnt is $m/\Delta t$ = 1300 kg/s so the thrust force F

F = 1300 kg/s × 50,000 m/s = 65,000,000 Nt

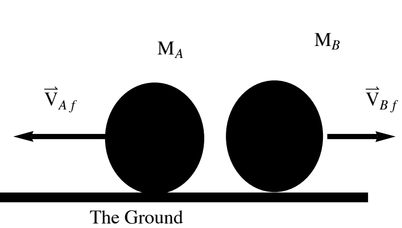
or converting, this is 15 million pounds of thrust which is large by any measure.

Conservation of Linear Momentum in a Collision:

Many collisions involve objects of nearly the same mass and in such cases, the velocities of both particles change as a result of the collision. The prototypical example is a moving billiard ball hitting another billiard ball at rest. (Ignore the fact that billiard balls roll because that causes an additional complication and we want to keep things simple.) The situation before the collision is pictured below:



The situation after the collision is given below and both billiard balls are assumed (for the moment) moving after the collision with velocity $\vec{V}_{Af}$ for ball A and velocity $\vec{V}_{Bf}$ for ball B:



Proceed by writing Newton's 2nd law for each mass.

Ball A: The momentum of M_A before the collision is $\vec{P}_{A0} = M_A \vec{V}_{A0}$ and

the momentum of mass M_A after the collision is $\vec{P}_{Af} = M_A \vec{V}_{Af}$.

Newton's 2nd law for mass M_A is

$\vec{F}_{B\to A} \Delta t = \vec{P}_{Af} - \vec{P}_{A0}$

where $\vec{F}_{B\to A}$ is the force of M_B on M_A during the collision. The two balls are in contact at time Δt.

Ball B: The momentum before the collision is $\vec{P}_{B0} = M_B \vec{V}_{B0}$ and

the momentum after the collision is $\vec{P}_{Bf} = M_B \vec{V}_{Bf}$.

Newton's 2nd law for mass M_B is

$\vec{F}_{A\to B} \Delta t = \vec{P}_{Bf} - \vec{P}_{B0}$

Combine: Adding the Newton's 2nd Law equations for mass M_A and mass M_B and obtain

$(\vec{F}_{B\to A} \Delta t + \vec{F}_{A\to B} \Delta t) = \vec{P}_{Af} - \vec{P}_{A0} + \vec{P}_{Bf} - \vec{P}_{B0}$

which looks like a mess. However, the left hand side of the above equation vanishes due to

Newton's 3rd Law $\vec{F}_{B\to A} = -\vec{F}_{A\to B}$ so we get more simply

$0 = \vec{P}_{Af} - \vec{P}_{A0} + \vec{P}_{Bf} - \vec{P}_{B0}$

and after a final rearrangement (all the initial momentum on the left side and final momentum on the right) we get

$\vec{P}_{A0} + \vec{P}_{B0} = \vec{P}_{Af} + \vec{P}_{Bf}$

This is called Conservation of Linear Momentum. Using the *definition* of Linear Momentum we can write Conservation of Linear Momentum explicitly for the billiard ball problem

$$M_A \vec{V}_{A0} + M_B \vec{V}_{B0} = M_A \vec{V}_{Af} + M_B \vec{V}_{Bf}$$

When this Law Holds: Linear Momentum is conserved if there are not external forces acting on the system and there are only internal forces that cancel out due to Newton's 3nd Law. Generally when there are external forces, Linear Momentum is NOT conserved. For example, when the ball hit the wall, the wall provided an external force on the ball.

Conservation of Momentum is different from Conservation of Energy which for this example is

$$KE_{A0} + KE_{B0} = KE_{Af} + KE_{Bf}$$

Using the explicit formulas for Kinetic Energy you get

$$\frac{1}{2} M_A V_{A0}{}^2 + \frac{1}{2} M_B V_{B0}{}^2 = \frac{1}{2} M_A V_{Af}{}^2 + \frac{1}{2} M_B V_{Bf}{}^2$$

for Conservation of Energy for this collision process.

Special Case: $M_A = M_B$ the Billiard Ball Problem

When the two masses are equal then ***the mass cancels out*** on both sides of the equations for conservation of linear momentum and conservation of kinetic energy thus

$$\vec{V}_{A0} + \vec{V}_{B0} = \vec{V}_{Af} + \vec{V}_{Bf}$$

$$V_{A0}^2 + V_{B0}^2 = V_{Af}^2 + V_{Bf}^2$$

Assuming ball B is at rest initially makes things even simpler

$$V_{A0} = V_{Af} + V_{Bf}$$

$$V_{A0}^2 = V_{Af}^2 + V_{Bf}^2$$

and the collision is only in one dimension so vectors are not involved. If you square the linear momentum equation you get

$$V_{A0}^2 = V_{Af}^2 + 2\, V_{Af}\, V_{Bf} + V_{Bf}^2$$

Comparison of the above equation with the energy equation indicates $V_{Af} V_{Bf} = 0$.

There are three possibilities: Either $V_{Af} = 0$ or/and $V_{Bf} = 0$.

Case #1: If $V_{Af} = 0$ then from the momentum equation $V_{Bf} = V_{A0}$. If you have every played billiards then you have run into this case where ball A hits another ball B which is initially at rest and ball A comes to a stop. All the velocity of ball A is transferred to the ball B .

Case #2: If $V_{Bf} = 0$ then from the momentum equation $V_{Af} = V_{A0}$. This case makes no sense physically since it means ball A passes through ball B and ball continues on with the original velocity it had. During this process, ball B remains a rest!? This never happens.

Case #3: $V_{Af} = 0$ AND $V_{Bf} = 0$ so both balls are at rest after the collision. But this would mean that the total final kinetic energy is zero. This would mean energy is not conserved (but originally we said assume KE is conserved) and this cannot be since initially mass A has kinetic energy.

Another Special Initial Condition: Billiard Ball Problem *(Again)*

Momentum Conservation: Suppose two balls are moving toward each with the same speed $V_{A0} = -V_{B0}$ and the initial momentum of ball A cancels the initial momentum of ball B. The momentum conservation equation becomes $0 = V_{Af} + V_{Bf}$ which implies that $V_{Bf} = -V_{Af}$

Energy Conservation: The kinetic energy equation becomes $V_{A0}{}^2 = V_{Af}{}^2$ because the A term is the same as the B term initially and similarly for after the collision.) . There are two possibilities from the kinetic energy conservation equation: $V_{Af} = -V_{A0}$ or $V_{Af} = V_{A0}$

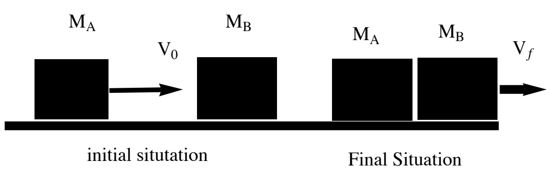
Case #1: The second case $V_{Af} = V_{A0}$ is never seen on a billiard table are anywhere else since this would mean ball A would pass through ball B and continue on with the same velocity!!

Case #2: The other case, $V_{Af} = -V_{A0}$ when combined with $V_{A0} = -V_{B0}$ this means that $V_{Af} = V_{B0}$ which means after the collision ball A has the same velocity as ball B did before the collision.

Also since $V_{Bf} = -V_{Af}$ combined with $V_{Af} = -V_{A0}$ means that $V_{Bf} = V_{A0}$. The effect of the collision is that ball A and ball B interchange their momentum.

Case where Momentum is Conserved but Kinetic Energy is Not Conserved

Suppose you imagine one block A sliding on a frictionless table toward block B which is initially at rest. As a result of the collision, the two mass stick together (think of Velcro) and move off together with velocity V_f.



Conservation of momentum for this problem is

$M_A V_0 = (M_A + M_B) V_f$

so we can solve for the final velocity V_f in terms of the two mass M_A and M_B and the initial velocity V_0 and obtain

$V_f = \frac{M_A V_0}{(M_A+M_B)}$

so we did NOT need conservation of energy to get the answer (which is a good thing--see below).

NUMERICAL EXAMPLE: Suppose M_A=M_B = 2 kg and V_0 = 5 m/s then

$V_f = \left(\frac{2\,\text{kg}}{2\,\text{kg} + 2\,\text{kg}}\right) 5\, m/s = 2.5\, m/s$

which is half the initial velocity of M_A .

What about energy conservation? Energy is NOT conserved in this collision.

Before the collision $KE_0 = \frac{1}{2} M_A V_0^2 = \frac{1}{2}\, 2\text{ kg} *(5\, m/s)^2 = 25$ Joules.

After the collision is $KE_f = \frac{1}{2} (M_A + M_B) V_f^2 = \frac{1}{2}\, 4\text{ kg} *(2.5\, m/s)^2 = 12.5$ Joules

$KE_0 \neq KE_f$

The lost 12.5 Joules of Kinetic Energy appears as heat which causes the two masses to slightly increase their temperature. Mechanical energy is NOT conserved. The total energy including heat energy is conserved.

Elastic and Inelastic Collisions:

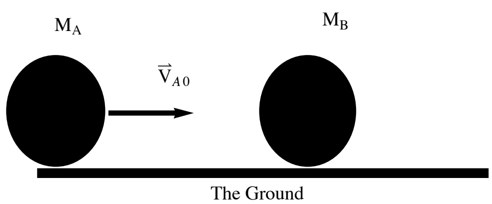
When Mechanical Energy is conserved we say the collision is "elastic" (think of billiard balls).

If the Mechanical Energy is NOT conserved in a collision is called "inelastic".

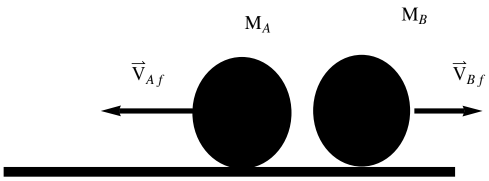
The total energy is always conserved. (Except for situations where mass is created or destroyed as in for example positron-electron annihilation. Even in cases like this, if you included the energy equivalent of mass via E=mc^2 then the total mass-energy is conserved.)

Case where one Mass is much Greater than the Other Mass

Suppose we have again the case where initially



and after the collision we have



Conservation of Linear Momentum explicitly for the billiard ball problem

$$M_A\ V_0 = M_A\ V_{A\,f} + M_B\ V_{B\,f}$$

and Conservation of Kinetic Energy

$$\frac{1}{2} M_A V_0^{\ 2} = \frac{1}{2} M_A V_{Af}^{\ 2} + \frac{1}{2} M_B V_{Bf}^{\ 2}$$

for this collision process. Only mass M_A was moving with velocity V_0 initially.

Suppose we know the initial velocity V_0 and the two masses M_A and M_B.

We have two equations, we can solve for the two unknowns: V_{Af} and V_{Bf} .

Start with the momentum conservation equation and write

$$V_{B\,f} = \frac{M_A}{M_B} V_0 - \frac{M_A}{M_B} V_{A\,f}$$

Next use this to eliminate in the Conservation of Energy equation obtaining

$$M_A V_0^{\,2} = M_A V_{Af}^{\,2} + M_B\left(\frac{M_A}{M_B} V_0 - \frac{M_A}{M_B} V_{Af}\right)^2$$

which looks like a mess but there now is only one unknown V_{Af}. But we can write more simply

$$M_A V_0^{\,2} = M_A V_{Af}^{\,2} + \frac{M_A^{\,2}}{M_B} (V_0 - V_{Af})^2$$

Expanding the binomial and collecting we get

$$M_A V_0^{\,2} = M_A V_{Af}^{\,2} + \frac{M_A^{\,2}}{M_B} (V_0^2 + V_{Af}^2 - 2 V_0 V_{Af})$$

which is a quadratic equation for V_{Af} so we write it in standard form

$$\left(M_A + \frac{M_A^{\,2}}{M_B}\right) V_{Af}^{\,2} - 2\,\frac{M_A^{\,2}}{M_B} V_0 V_{Af} + \left(\frac{M_A^{\,2}}{M_B} - M_A\right) V_0^2 = 0$$

or a little more simply as

$$\left(1 + \frac{M_A}{M_B}\right) V_{Af}^{\,2} - 2\,\frac{M_A}{M_B} V_0 V_{Af} + \left(\frac{M_A}{M_B} - 1\right) V_0^2 = 0$$

Special Case: $M_A << M_B$ which is the case of a pea (M_A) hitting a battleship (M_B).

$\frac{M_A}{M_B}$<<1 and terms of

$$\left(1 + \frac{M_A}{M_B}\right) V_{Af}^{\,2} - 2\,\frac{M_A}{M_B} V_0 V_{Af} + \left(\frac{M_A}{M_B} - 1\right) V_0^2 = 0$$

containing $\frac{M_A}{M_B}$ can be dropped so we get simply

$$V_{Af}^{\,2} - V_0^2 = 0 \quad \text{or} \quad V_{Af}^{\,2} = V_0^2$$

There are two solutions: $V_{Af} = - V_0$ or $V_{Af} = V_0$ but only the minus root makes physical sense

$$V_{Af} = -V_0$$

Using the above in

$V_{Bf} = \frac{M_A}{M_B} V_0 - \frac{M_A}{M_B} V_{Af}$

we get $V_{Bf} =$ 0 to a good approximation so the battleship does not move (like you couldn't guess that!) and pea just bounces of with the same speed it came in with.

Special Case: $M_A >> M_B$ which is the case of a bowling ball (M_A) hitting a pea (M_B).

$\frac{M_A}{M_B}$>>1. Now the terms of

$$\left(1 + \frac{M_A}{M_B}\right) V_{Af}{}^2 - 2 \frac{M_A}{M_B} V_0 V_{Af} + \left(\frac{M_A}{M_B} - 1\right) V_0{}^2 = 0$$

containing $\frac{M_A}{M_B}$ are dominant so we get more simply

$$\left(\frac{M_A}{M_B}\right) V_{Af}{}^2 - 2 \frac{M_A}{M_B} V_0 V_{Af} + \left(\frac{M_A}{M_B}\right) V_0{}^2 = 0$$

and thus after cancellation of the masses

$$V_{Af}{}^2 - 2 V_0 V_{Af} + V_0{}^2 = 0$$

Solving the quadratic equation we get finally just one root

$$V_{Af} = \frac{2 V_0 \pm \sqrt{4 V_0{}^2 - 4 V_0{}^2}}{2} = V_0$$

To a good approximation the bowling ball is not effected by hitting the pea which is hardly a surprise. The pea's final velocity is zero

$V_{Bf} = \frac{M_A}{M_B} V_0 - \frac{M_A}{M_B} V_{Af} = 0$

after the collision.

More General Case: For two equal masses $M_A = M_B$

$$\left(1 + \frac{M_A}{M_B}\right) V_{Af}^{\ 2} - 2\,\frac{M_A}{M_B}\, V_0\, V_{Af} + \left(\frac{M_A}{M_B} - 1\right) V_0^{\ 2} = 0$$

becomes much more simple

$$2\, V_{Af}^{\ 2} - 2\, V_0\, V_{Af} = 0$$

which can be written

$$V_{Af}(V_{Af} - V_0) = 0$$

which has two solutions $V_{Af}=0$ and $V_{Af} = V_0$. Finally use these two cases in $V_{Bf} = \frac{M_A}{M_B} V_0 - \frac{M_A}{M_B} V_{Af}$ which for $M_A = M_B$ becomes $V_{Bf} = V_0 - V_{Af}$

Case #1: $V_{Af}=0$ Ball A stops and Ball B moves with speed $V_{Bf} = V_0$

Case #2: $V_{Af} = V_0$ Ball A continues on as though it passed through Ball B and $V_{Bf} = 0$ which makes no sense so this case is ignored on physical grounds.

Case of Ballistic Pendulum V_f

What is it? A ballistic pendulum is a device used to measure the velocity of a bullet from a gun.

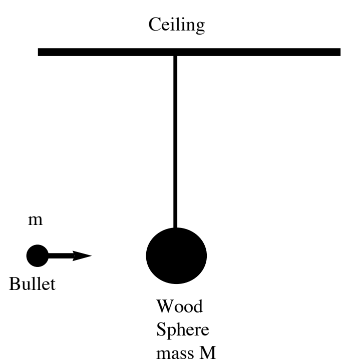
How does it work?

1. A bullet is fired into a wooden ball attached by a string to the ceiling as pictured below.

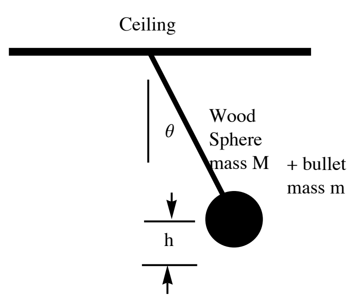
2. The bullet comes to rest with respect to the wood sphere but the bullet and sphere have a velocity as a result of the bullet's impact.

3. The sphere including the bullet then swings upward a maximum height h. Using h the velocity of the bullet can be obtained.

Before the Bullet Hits Sphere



After the Bullet has Hit the Sphere



1. The collision of the bullet (of mass m) with the wood ball (of mass M) is inelastic (energy is NOT conserved) but momentum is conserved in this collision.

2. The initial velocity of the bullet is V_0 (this is what we want to find).

3. After the collision, the Wood and bullet have a much smaller velocity V_f.

4. Conservation of Linear Momentum gives for the inelastic collision of the bullet with the wood

m V_0 = (m+M) V_f.

5. After the collision, the bullet and wood sphere rise up a height h above the original location of the sphere.

6. Energy is conserved for this part of the process so the kinetic energy $\frac{1}{2}(m+M)V_f^2$ gets converted into gravitational potential energy (m+M)g h

$\frac{1}{2}(m+M)V_f^2 = (m+M)g\,h$

7. After cancellation of the (m+M) we get

$V_f = \sqrt{2\,g\,h}$

8. Using this to eliminate V_f in the momentum conservation equation yields

$V_0 = (1+\frac{M}{m})\ V_f = (1+\frac{M}{m})\sqrt{2\,g\,h}$

You know the bullet mass m and the mass of the wood sphere M. You measure the height h after firing the bullet and you can find the initial velocity V_0 of the bullet.

Numerical Example: Suppose the mass of the bullet m is 3 gm=0.003 kg and the block of wood has mass M=2.0 kg and block of wood rises up h=0.25 m. Find the velocity of the bullet as it leaves the gun barrel V_0

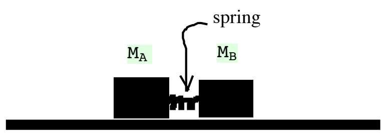
In[151]:= $V_0 = \left(1 + \frac{2.0}{0.003}\right) * \sqrt{2 * 9.8 * 0.25}$

Out[151]= 1477.94

The velocity of the bullet is 1,478 m/s as it leaves the gun barrel (neglecting air resistance which would slow the bullet down before it hit the wood.

Case of Exploding System of Two Masses

Suppose two blocks are initially at rest and tied together and with a compressed spring between the two blocks.



Initial State

Final State

The total momentum is zero initially thus conservation of momentum yields

$0 = M_A\ \vec{V}_{A\,f} + M_B\ \vec{V}_{B\,f}$

and solving $\vec{V}_{A\,f}$ we get

$\vec{V}_{A\,f} = -\frac{M_B}{M_A}\ \vec{V}_{B\,f}$

Special Cases:

Case #1: $M_A = M_B$

$\vec{V}_{A\,f} = -\vec{V}_{B\,f}$

The two masses have equal and opposite velocities.

Case #2: $M_A \gg M_B$ so that $\frac{M_B}{M_A}$<<1 (e.g. A is a gun and B is a bullet)

$\vec{V}_{A\,f} \ll 1$

Regardless of what $\vec{V}_{B\,f}$ is, it follows from $V_{A\,f} = -\frac{M_B}{M_A}\ V_{B\,f}$ that $\vec{V}_{A\,f}$ is much smaller.

Numerical Example: Suppose the mass of the bullet M_B is 3 gm=0.003 kg and the gun has mass M_A=2.0 kg then $\frac{M_B}{M_A}$=0.0015 and suppose velocity of the bullet is $V_{B\,f}$ = 1478 m/s as in the previous example, then the velocity of the gun is

$$\frac{M_B}{M_A} = \frac{0.003}{2}$$

Out[152]= 0.0015

In[153]:= $V_{A\,f}$ = 0.0015 * 1478

Out[153]= 2.217

So the gun's velocity is 2.2 m/s which is much smaller than the bullet's velocity.