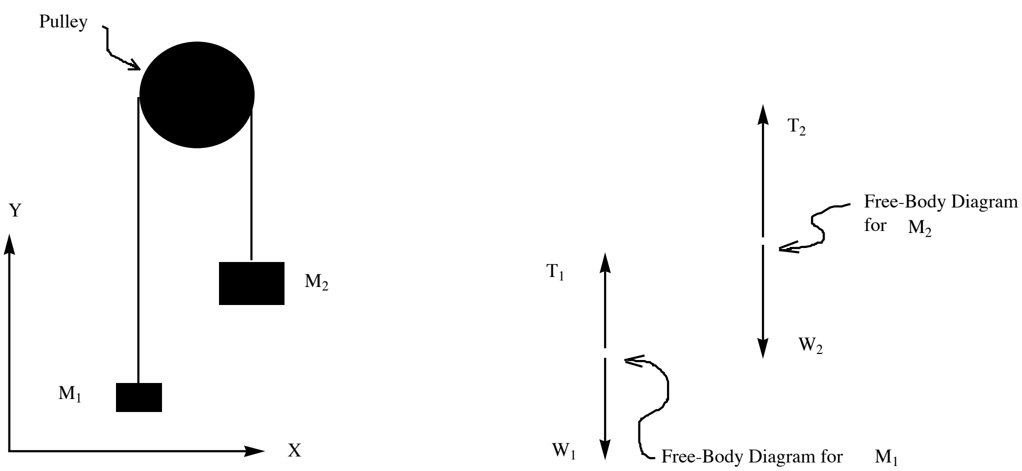
7. Newton's 2nd Law in More Complicated Problems and Friction

The Atwood's Machine is used below to help in understanding how Newton's 2nd law applies to a system of two connected masses. It is not a practical machine.

Suppose two different masses M_1 and M_2 are attached to a rope which is placed over a pulley as indicated in the diagram below. There are TWO free-body diagrams since there are two masses in this problem.



i) W_1 is the force of gravity on mass M_1 and W_2 is the force of gravity on mass M_2. $W_1 = M_1 g$ and $W_2 = M_2 g$.

ii) T_1 is the tension force on M_1 due to the attached rope and T_2 is the tension force on M_2 due to the attached rope.

iii) Assume an ideal rope that is a mass-less rope that does not stretch so $T_1=T_2$.

iv) The only motion is in the y-direction so we only need to consider Newton's 2nd law in the form $F_y = M\, a_y$.

Since $M_2 > M_1$ the pulley will rotate clockwise.

v) **Key Idea**: Apply Newton's 2nd law to each mass separately.

***For Mass* M_1**: $F_1=M_1a_1$ where the total force on M_1 in the y-direction is $F_1=T_1-W_1$ and a_1 is the acceleration of mass M_1 in the upward or positive y-direction (a_1 is a positive number). **eqt. #1**: $T_1-W_1 = M_1a_1$ (T_1 is a positive number.)

***For Mass* M_2**: $F_2=M_2 a_2$ where the total force on M_2 in the y-direction is $F_2=T_2-W_2$. Since the acceleration of M_2 is downward, the acceleration has a negative y-component ($-a_2$) where a_2 is a positive number.

eqt. #2: $T_2-W_2 = M_2(-a_2)$ (T_2 is a positive number.)

Ideal Rope: The rope does not stretch and if M_1 moves a distance then M_2 moves an equal distance so $a_1 = a_2 = a$ and

Ideal Pulley: Mass-less and only serves change the direction of the tension force in the rope so $T_1 = T_2 = T$.

Equations #1 and #2 simplify to

$T-W_1 = M_1a$ and

$T-W_2 = M_2(-a)$

Subtract equation #2 from equation #1 and obtain

$W_2 - W_1 = (M_1 + M_2)a$

Notice the tension T canceled out. Finally solving for the acceleration a you get $a = \frac{W_2 - W_1}{(M_1 + M_2)} = \frac{(M_2 - M_1)\, g}{(M_1 + M_2)}$.

1. Notice the effective inertia of the combined systems is the sum of the two masses $(M_1 + M_2)$.

2. The net gravitational force on the two masses is due to the difference of the two masses $(M_2 - M_1)\, g$.

A Numerical Example: Suppose M_1 =3 kg and $M_2 = 7$ kg. Determine the acceleration of the masses.

```
M1 = 3.;  M2 = 7.;  g = 9.8;
     (M2 - M1) * g
a = ---------------
        M1 + M2
3.92
```

So the acceleration is a=3.9 m/s^2 and $a_1 = a_2 = a$.

Atwood's Machine Experiment on the World Wide Web: Rutgers University

Go to the web address http://paer.rutgers.edu/PT3/experiment.php?topicid=8&exptid=4 for a QuickTime video atwood.mov of the Atwood's machine which you may play through Safari or Explorer on your computer at home. Here M_2=200 gm and M_1=150 gm. Compute the acceleration using (note gm are converted into kg)

```
M1 = 150. / 1000;  M2 = 200. / 1000;  g = 9.8;
     (M2 - M1) * g
a = ---------------
        M1 + M2
1.4
```

$a=1.4\ m/s^2$ The masses are released from rest and observe how long it takes M_2 to travel y=0.5 m (M_2 starts out at 20 cm so observe the time it takes for M_2 to get to 70 cm).

$\frac{(70\ \text{cm} - 20\ \text{cm})}{100\ \text{cm}/m} = 0.5\ m$ If you use the kinmatic equation $y = \frac{1}{2} a\, t^2$ and solve for the time t it takes M_2 to travel y = 0.5 m you get

```
y = 0.5;
t = Sqrt[2 * y / a]
0.845154
```

So it should take M_2 t = 0.845 sec to travel y = 0.5 m. The QuickTime has a frame rate of 30 fps where **fps** is **frames per second**. The inverse of this is second per frame Δt which works about to be

```
Δt = 1 / 30.
0.0333333
```

The number of frames Num it takes M_2 to travel y = 0.5 m which is t = 0.845 sec is given by

```
Num = t / Δt
25.3546
```

which is about half way between 25 frames and 26 frames provided Newton's 2nd Law is valid for the Atwood's machine. This should be a bit of good evidence for you that Newton's 2nd law is true but of course many more values of M_2 and M_1 need to be tried before you really should be convinced. In fact, to be convincing you should work out Newton's 2nd law for all possible examples not just the Atwood's machine but also, for example a block sliding down an inclined plane. But of course it is impossible to work out all possible examples and so you can never really prove Newton's 2nd Law is universally true.

The Force Due to Friction f: Friction is a force between two objects in relative motion. Friction is due to intermolecular forces between the atoms of the to solid objects.

When we refer to friction, we mean the force of resistance between two solid object. "Air resistance" or "drag force" has some different properties and will be discussed elsewhere.

Properties of the Friction Force f:

1. f acts in a direction opposite the direction of motion.

2. The magnitude of f is given by

$f = \mu$ N

where μ is called the coefficient of friction and N is the normal force discussed previously.

3. μ is a number having no dimension in the range $0 \leq \mu \leq 1$. μ depends upon the materials involved.

4. Actually there are at least two different coefficients of friction: "static" friction μ_s and "kinetic" friction μ_k. The concepts are similar but satisfy

different equations: $f_s = \mu_s$ N and $f_k = \mu_k$ N. f_s applies before motion begins and f_k after motion starts.

5. μ_s and μ_k depend on the materials involved. For example, for example for wood on wood μ_s=0.4 and $\mu_k = 0.2$ For Teflon on Teflon μ_s=μ_k=0.04 μ is smaller when the friction force is less. See table 4.2 for other examples.

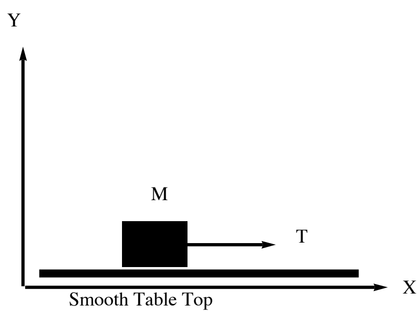
6. μ_s is the static coefficient of friction that exists before motion takes place and μ_k is the kinetic coefficient of friction after relative motion between the block and the table top begins.

7. $\mu_s \geqslant \mu_k$ usually and the applied force has to be greater than the static friction force f_s for motion to occur. Once motion begins, the kinetic friction force f_k applies.

8. There is also coefficient of rolling friction for cylinders in contact with a surface but this will not be discussed in this course.

2. **Example of a Block Pulled by a Rope with Friction Involved**:

This is similar to a problem #3 treated in the last lecture except now friction $f = \mu$ N is included. Assume M=6 kg and T=2 Nt .

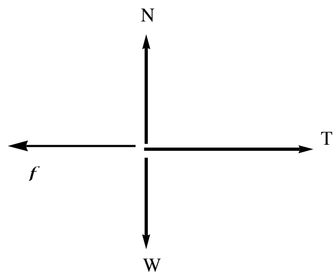


i) The X, Y coordinate system (or frame of reference) used for this problem is indicated.

ii) There is friction $f_s = \mu_s$ N with $\mu_s = 0.4$ if there is no relative motion between the wooden mass M and the wood table top. If there is motion, then $f_k = \mu_k$ N with $\mu_k = 0.2$

iii) The force in the rope is called a Tension and is labeled T.

iv) The free-body diagram for this problem appears below:



v) The force of gravity W, the normal force N, and tension T were discussed before. Only the friction force f is new.

vi) Newton's 2nd law $\vec{F} = M\,\vec{a}$ is actually two equations:

$F_x = M\,a_x$ and $F_y = M\,a_y$.

vii) F_y is the total force in the y-direction and in this example $F_y = N - W = 0$ assuming there is no movement in the y-direction. For our example,

$N = W = Mg = 6\text{ kg} \times 9.8\text{ m}/s^2 = 58.7\text{ Nt}$.

```
6 * 9.8
58.8
```

viii) The total force in the x-direction is F_x. For motion to occur, $F_x = T - f_s > 0$. In other words, if $T > f_s$ then $F_x > 0$ and motion occurs. T= 2 Nt and $f_s = \mu_s$ N=0.4×58.7 Nt = 23.5 N

so it follows that $T < f_s$ in this example so motion does not occur in the x-direction.

```
0.4 * 58.7
23.48
```

ix) Suppose we increase the tension T in the rope so motion does occur in the x-direction. Let us assume for example, T=26 Nt then $T > f_s$ and motion in the x-direction begins. However, the motion is now determined by f_k and NOT f_s.

x) Newton's 2nd law in the x-direction yields $T - f_k = M a_x$ and solving for the acceleration yields $a_x = \frac{(T - f_k)}{M}$. Notice that if the object is moving, then we use f_k and NOT f_s .

A Numerical Example: We already determined that the tension T is large enough so that motion in the x-direction occurs. The frictional force to use for the motion is $f_k = \mu_k$ N=0.2×58.7 Nt = 11.7 Nt. Since T=26 and f_k=11.7Nt this is consistent with $T > f_k$ and motion to occur. The acceleration a_x in the x-direction is

```
0.2 * 58.7
11.74
```

$a_x = \frac{(T - f_k)}{M} = \frac{(26\text{ Nt} - 11.7\text{ Nt})}{6\text{ kg}}$ = 2.4 m/s^2.

$$\frac{26 - 11.7}{6}$$

```
2.38333
```

Question: What would be the acceleration if there were no friction?

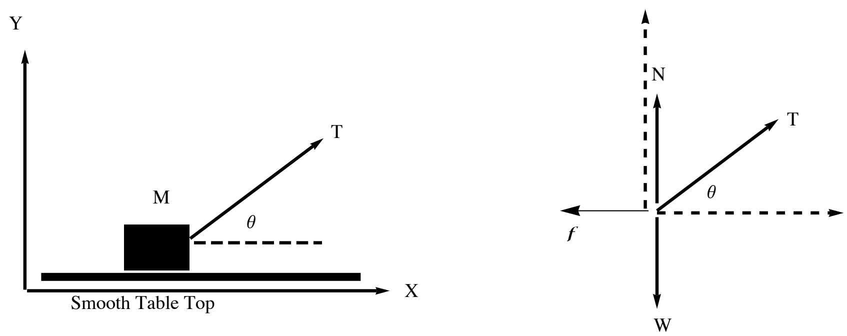
Answer: $a_x = \frac{T}{M} = \frac{26\text{ Nt}}{6\text{ kg}}$ = 4.3 m/s^2 and this is greater than with friction.

```
26. / 6
4.33333
```

◀ | ▶

Example: **A block pulled by a rope**. (Ex #5 lecture #6)

The rope pulling the block has a tension force T and is at an angle θ above the horizontal. Here there is friction between the block and the table top surface.



The free-body diagram involving the forces is on the right.

i) Newton's 2nd law appears as before in component form

$F_x = M\,a_x$ and $F_y = M\,a_y$.

ii) The tension has two components now:

T Cos[θ] is the x-component and

T Sin[θ] is the y-component.

iii) F_x = T Cos[θ]- f_s is the total force in the x-direction.

F_y = T Sin[θ]+N-W is the total force in the y-direction.

Assume the block does NOT move in the y-direction (in other words the block stays on the table) so it follows that
the net force in the y-direction is zero, that is T Sin[θ]+N-W = 0 The normal force N (which is necessary to determine the friction force) can be obtained thus

N= W - T Sin[θ]

Aside: If the tension T in the rope is large enough, the rope will pull the block off of the table. This happens if
T Sin[θ] > W and there is no nomal force N=0. The net force in the y-direction is positive and as a result there is a positive acceleration in the y direction. Friction obviously is not involved since the block is not on the table so this is not a very interesting case. When you are given a problem to solve, whether or not there is motion in the y-direction should be spelled out.

The x-Motion: If the block is to move in the x-direction, then the net force in the x-direction must be positive, that is $F_x > 0$.

Since the net force in the x-direction is $F_x = T \, \text{Cos}[\theta] - f_s$ it follows

$T \, \text{Cos}[\theta] > f_s$ for the block to move.

Example: Assume M=6 kg, T=25 Nt, θ=30°, $\mu_s = 0.4$, and $\mu_k = 0.2$ The normal force is given by (assuming a_y=0)

$$N = W - T \, \text{Sin}[\theta] = 6 \text{ kg} \times 9.8 \text{ m}/s^2 - 25 \text{ Nt} \times \text{Sin}[30°]$$

```
6 * 9.8 - 25 * Sin[30 °]
46.3
```

so the Normal force N=46.3 Nt and since this is positive this is consistent with the bock staying on the table. Next compute the total force in the x-direction using the static coefficient of friction μ_s (assuming for the moment there is not x-motion).

$$F_x = T \, \text{Cos}[\theta] - f_s = T \, \text{Cos}[\theta] - \mu_s N$$

```
25 * Cos[30 °] - 0.4 * 46.3
3.13064
```

so F_x=3.1 Nt and it is positive, $F_x > 0$ so there is motion in the positive x direction with the acceleration a_x given by using the kinetic coefficint of friction $f_k = \mu_k N$

$$a_x = \frac{F_x}{M} = \frac{T \, \text{Cos}[\theta] - f_k}{M} = \frac{T \, \text{Cos}[\theta] - \mu_k N}{M} = \frac{T \, \text{Cos}[\theta] - \mu_k (W - T \, \text{Sin}[\theta])}{M}$$

```
ax = (25 * Cos[30 °] - 0.2 * 46.3) / 6
2.06511
```

So the acceleration in the x-direction is a_x = 2.06 m/s^2.

The behavior a_x of as a function of θ is a little complicated. When $\theta$is small for example, θ=0 the effect of the Tension in T Cos[θ] is the largest and this tends to increase a_x . When θ is small, T Sin[θ] is small and the normal force N is the greates and the fricition force has its largest effect reducing the acceleration in the x-direciton.

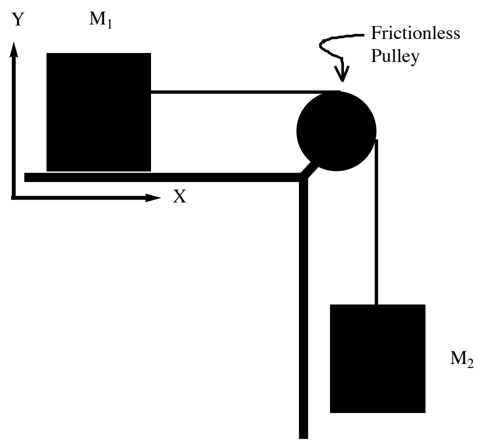
If there were no friction, then the acceleration a_x would be given by

$$a_x = \frac{F_x}{M} = \frac{T\,\mathrm{Cos}[\theta]}{M}$$

which works out to be a_x = 3.6 m/s^2 which is larger a larger acceleration than with friction as it should be intuitively.

```
ax = 25. * Cos[30 °] / 6
3.60844
```

Example: Perhaps the simplest apparatus to verify Newton's 2nd Law has the form. Two masses M_1 and M_2 connected by a rope over a pulley as indicated by the diagram below:



Free-Body Diagram M_1 Free-Body Diagram M_2

Notation: F_1^x is the force on M_1 in the x-direction and a_1^x is the acceleration of M_1 in the x-direction and a_1^y=0 since the block stays on the table top. The motion of M_2 is only in the negative y-direction so for $a_2^x = 0$ and the acceleration in the y-didrection $-a_2^y$ where a_2^y is **positive**.

Newton's Laws for Mass M_1:

$F_1^x = M_1 a_1^x$ and with $F_1^x = T_1$ becomes $T_1 = M_1 a_1^x$ equation (1)

$F_1^y = M_1 a_1^y$ together with $F_1^y = N_1 - W_1$ and $a_1^y = 0$ becomes

$N_1 - W_1 = 0$ and thus $N_1 = W_1 = M_1 g$

Newton's Laws for Mass M_2:

$F_2^x = M_2 a_2^x$ with $F_2^x = 0$ and $a_2^x = 0$ since there is no motion in x-direction.

$F_2^y = M_2 a_2^y$ together with $F_2^y = T_2 - W_2$ yields

$T_2 - W_2 = M_2 (-a_2^y)$ equation (2)

where a_2^y is **positive**. Subtract equation (2) from equation (1) and obtain

equation (3) $T_1 - T_2 + W_2 = M_1 a_1^x + M_2 a_2^y$.

The pulley is ideal (it no mass or moment of inertia) so $T_1 = T_2 = T$ and the

rope is ideal (does not stretch) so $a_1^x = a_2^y = a$.

Thus the tensions cancel out and the acceleration terms are combined

$W_2 = (M_1 + M_2)\, a$ and solving for the acceleration

$$a = \frac{W_2}{(M_1 + M_2)}\ .$$

Observations:

1. The inertia of the system is the sum of the masses $(M_1 + M_2)$ just like in the Atwood's machine.

2. The force causing the motion is $W_2 = M_2\, g$. This is different from the Atwood's machine.

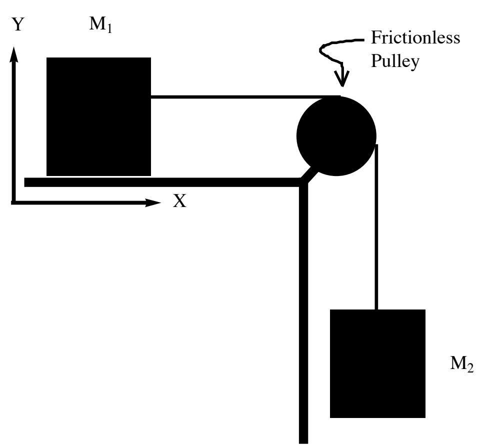
3. This apparatus is the most easy to work with to verify Newont's 2nd law since changing M_2 has an effect on the net force moving the two masses. The only cost is that changing M_2 also effects the total inertia $(M_1 + M_2)$ of the system .

A Numerical Example: Suppose M_1 =3 kg and $M_2 = 7$ kg which is the same as in the original Atwood's machine. Determine the acceleration of the masses.

```
M1 = 3.; M2 = 7.; g = 9.8;
a = M2 * g / (M1 + M2)
6.86
```

So the acceleration is a=6.9 m/s^2 and $a_1 = a_2 = a$. Recall that in the original Atwood's machine a=3.9 m/s^2 which is less. Does this make inuitive sense to you?

WORKSHOP PROBLEM: Do the previous problem but include friction between the table top and the moving mass so M_1 =3 kg and $M_2 = 7$ kg. The friction force is $f_s = \mu_s$ N with $\mu_s = 0.4$ if M_1 is not in motion. If M_1 is in motion, then $f_k = \mu_k$ N with $\mu_k = 0.2$



Procedure:

1. Draw a free-body diagram for M_1 and another free-body diagram for M_2 .

2. Write Newton's 2nd law for each mass: You get two equations.

The equation for M_2 is the same as for the above problem.

The equation for M_1 is almost the same as for the above problem except now friction is included.

3. First assume the friction force is small enough so that the weight M_2 in able to move M_1 and use μ_k.

4. Write the equations for an ideal pulley and an ideal rope.

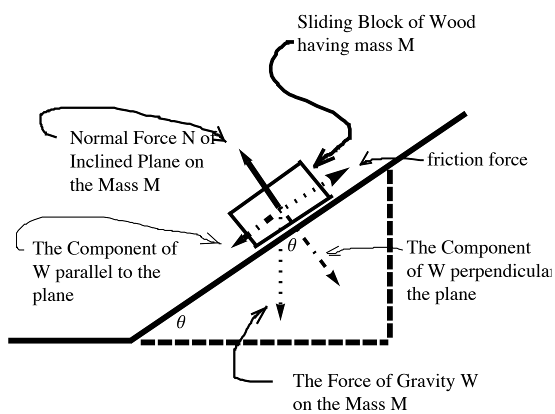
5. Combine the two equations of step 2. Use the ideal pulley and ideal rope relations.

6. Solve for the accelerations.

7. Is it possible for the friction to be large enough so there is no motion? If so, what is the condition for this? Do you use μ_s or μ_k in this condition?

Example: Motion on an Inclined Plane

Another example will be used to make sure you understand the above concepts. Consider a block of wood sliding down the top of an inclined plane as indicated below.



The forces acting on the mass M:

1. The force of gravity W=Mg acting on the block is downward toward the Earth's center.

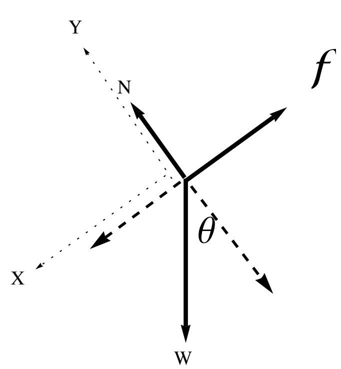
W has two components: One component parallel to the inclinded plane and the other component perpendicular to the inclinded plane.

2. N is the normal force of the inclined plane on the mass M. N is perpendicular the inclined plane.

3. There is also the force of friction $f = \mu$ N where μ is either the static or kinetic coefficient of friction depending on whether or not the mass M is moving.

There are basically three forces acting on the mass M: friction f, gravity W, and the normal force N.

The Free-Body Diagram: The solid lines are the three forces and the dashed lines are the components of W.



APPLICATION OF NEWTON'S 2ND LAW:

1. The X, Y coordinate system indicated on the diagram above is perhaps the simplest one to use

since the only motion is in the X direction and there is no motion in the Y direction.

2. Since there is no motion in the Y direction, the acceleration $a_y = 0$ and as a result of Newton's 2nd law, $F_y = 0$. There are two components of force in the Y directions: the normal force N and the y-component of the force of gravity - W Cos[θ]. The sum of these forces is zero: N - W Cos[θ] = 0 where W=Mg, In other words N = W Cos[θ]

3. There two forces in the X direction: (1) the friction force - f in the negative X direction with f=μ N a positive number and (2) the X-component of the force of gravity W Sin[θ]. Newton's 2nd law in the X direction appears W Sin [θ] - f = M a_x or solving for the acceleration

$$a_x = \frac{W\,\text{Sin}[\theta] - f}{M} = \frac{W\,\text{Sin}[\theta] - \mu\,N}{M} = \frac{W\,\text{Sin}[\theta] - \mu\,W\,\text{Cos}[\theta]}{M} = \frac{W\,(\text{Sin}[\theta] - \mu\,\,\text{Cos}[\theta])}{M} = g\,(\text{Sin}[\theta] - \mu\,\,\text{Cos}[\theta])$$

This a relatively simple result a_x = g (Sin[θ] - μ Cos[θ]) but it was based on the assumption that the friction force is small enough so that the block slides down the plane.

A Numerical Example: Suppose M=3 kg, θ=43°, $\mu_s = 0.40$, and $\mu_k = 0.20$ What is the acceleration of the mass down the plane? First compute (Sin[θ] - μ_s Cos[θ]) since this must be positive for the block to move.

```
θ = 43 °;
μs = 0.4;
Sin[θ] - μs * Cos[θ]
```

```
0.389457
```

Since we have a positive number we know the block will slide down the plane and we compute a_x using

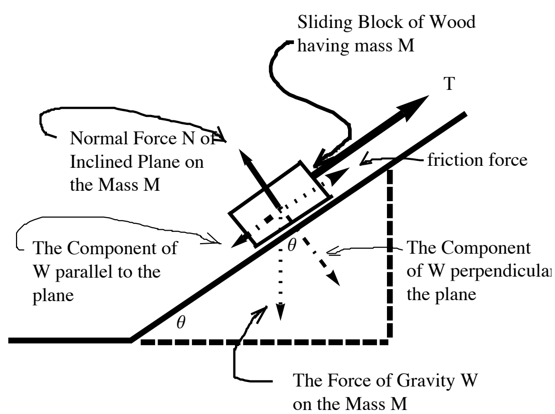
```
g = 9.8;
μk = 0.2;
ax = g * (Sin[θ] - μk * Cos[θ])
```

```
5.25013
```

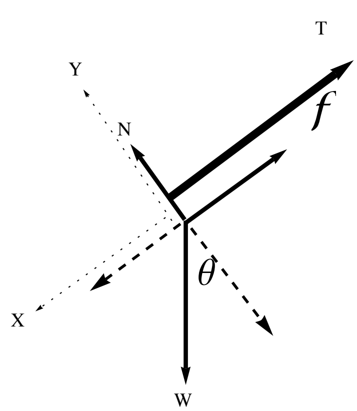
So the acceleration down the plane is a_x=5.25 m/s^2.

Another Example:

Suppose you have the apparatus of the previous problem but additionally a rope tied to the block of mass M and a tension T exerted on the block parallel the inclined plane and upward . So the problem appears like the diagram below:



The free-body diagram is



What heppens depends upon the size of the tension T in the rope. If the tension T is large enough, then the block moves up the inclined plane. If the tension is smaller, the block slides down the inclined plane but iwth an acceleration smaller than if T is present. Make sure you can solve this problem for the acceleration parallel to the inclined plane.