8. Newton's Law of Gravitation

Introduction and Summary

There is one other major law due to Newton that will be used in this course and this is his famous **Law of Universal Gravitation**. It deals with the force between any two massive objects. We will use the Law of Universal Gravitation together with Newton's Laws of Motion to discuss a variety of problems involving the motion of large objects like the Earth moving in orbit about the Sun as well as small objects like the famous apple falling from a tree. Also it will be shown that Newton's 3rd Law of Motion follows as a consequence of Newton's Law of Universal Gravitation.

Some additional topics that related to Newton's 1st and 2nd Laws of Motion also will be discussed. In particular, the concept of the **Non-Inertia Reference Frame** will be introduced and why it is useful. Also it will be shown how Newton's 1st Law and 2nd Law are NOT valid in Non-Inertial Reference Frames. It will be shown what is necessary to "fix-up" Newton's 2nd Law so it works in Non-Inertial Reference Frames. In particular, the examples of accelerated cars and elevators are used to illustrate the concept of the Non-Inertial Frame.

The Four Fundamental Forces of Nature:

There are **four basic or fundamental forces** known in nature. These forces are **gravity**, **electromagnetism**, **the weak nuclear force**, and **the strong nuclear force**. The force of gravity is described by Newton's Law of gravitation and the more recent modification called General

Relativity due to Einstein. Any two masses in the Universe feel the force of gravity and the gravitational force has the longest range of interaction of the known forces.

The electromagnetic force will be studied next semester and it involves the electric force which you know of as static electricity and the magnetic force which is involved in the operation of the compass. There are two kinds of charge called positive and negative by convention and the theory tells how to calculate the forces on a charge at rest or moving. The electric and magnetic forces are two sides of the same coin and combined theory is called electromagnetism and this unification was achieved by Maxwell in the late 1880's.

The strong nuclear force is the "glue" that holds an atomic nucleus together and it is also known that heavy particles or "Hadrons" like protons and neutrons actually consist of "Quarks" held together by "Gluons". Nuclei consist of protons which are positively charged and neutrons which are uncharged. All the positive peorona repel each other in the nucleus due to the electric force which is repulsive between like charges. If that were all there were to it, the nucleus would be unstable (that is blow apart). But at short distances the attractive strong nuclear force overpowers the repulsive electric force and the nucleus is more or less stable. (Less stable for radioactive atoms.)

The weak nuclear force is responsible for processes in which a Beta particle (an Electron) is produced among other things. Neutrons are slightly heavier than protons and neutrons can transform into protons spontaneously producing an electron and a neutrinos. The neutrino has almost zero mass and the neutrino has zero charge.

The "early Universe" 13 billion years ago was a very hot place and this corresponds to very high energy or the particles were moving very rapidly and having collisons frequently and violently. A very short time after the "Big Bang" of creation, the electromagnetic and the weak nuclear force were combined into one force called the Electro-Weak force because the collisons were so violent. However, for most experiments on Earth, the electromagnetic and weak forces are separate forces because the energy of collison involved is not enough to combine them into one force. The theory of this unification of forces was described by Weinberg, Glashow and Salam in the 1970's and has been experimentally verified.

Also it is believed that the strong nuclear force is also just a part of a theory now called Quantum Chromodynamics" which unifies the weak and strong nuclear forces as well as the electromagnetic force at even higher energy or even further back to the big bang beginning. This theory is now being tested at the new particle accelerator at CERN in Europe. One of the major particles they hope to observe at CERN is the socalled "Higgs Boson" which is expected to be much more massive than a proton and theory says the Higgs particle plays an important role in determining the masses of all the fundamenetal particles like protons, neutrons, and electrons. The Higgs particle has been called the "God particle" by some because of this important role. The odd force out is Gravitation but there is a somewhat speculative "String Theory" which claims to be a Theory of Everything that is all the forces and you can read about in a number of popular books.

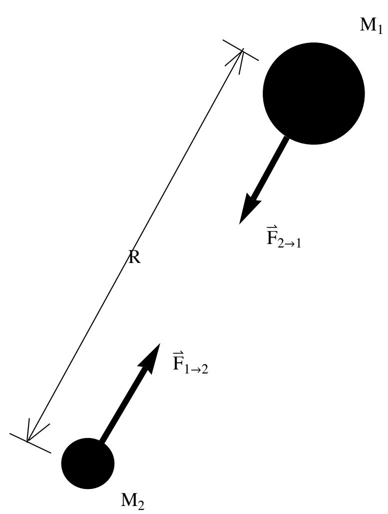
Here the focus is on Newton's original Law of Gravitation. All other forces in nature like, for example, friction or chemical forces can be explained in terms of these four fundamental forces of nature. However, there are some major mysteries left in physics. The galaxies do not appear to have enough visible mass in the form of stars to be stable. That is, if you take the observed rotation rate of galaxies, there is not enough gravitational glue to hold them together and the galaxies should fly apart. It is thought that only something like 10% of the particles in the universe consists of particles that have been observed in the laboratory on Earth like protons, neutrons, and electrons. Also the Big Bang Theory of the Universe seems to predict there is more matter in the Universe than is actually observed and the rest of the unseen matter is called "dark matter". What "dark matter" is currently a mystery. Another mystery is that the universe not only seems to be expanding but additionally the rate of expansion is increasing. This blowing up faster as time evolves seems to indicate there is at least one more additional force in the universe that opposes gravity and this is called "dark energy". Little is known about what this is and how it operates.

Newton's Law of Universal Gravitation:

This law describes the force between any two objects one having mass M_1 and the other having mass M_2. The force between the two masses is equal to the product of the two masses M_1 and M_2 and inversely proportional to the distance R between the masses squared

$$F = G \frac{M_1 M_2}{R^2} \tag{1}$$

G is a proportionality constant which is very small if S.I. units are involved G=6.67 $\times 10^{-11}$ $\frac{\text{Nt}-m^2}{\text{kg}^2}$. G is called the universal gravitation constant and the units of G are such that the m^2 in the denominator on the right hand side cancel out, the kg^2 in the numerator cancel, and the result should a force in Newtons. This force is thought to apply to any two masses whether two planets or an apple and the Earth so the law is called **Universal.** The gravitational force is said to be attractive and directed along the line between the two masses. A schematic of the situation is below. The force of mass 1 on mass 2 is indicated by $\vec{F}_{1\to 2}$ and force of mass 2 on mass 1 is indicated by $\vec{F}_{2\to 1}$. The lengths of the two force vectors are equal even if the masses are not equal and this fact follows from equation (1). Also, the force of mass 1 on mass 2 is $\vec{F}_{1\to 2}$ is **attractive** which means it is from mass directed toward mass 1 and similarly for $\vec{F}_{2\to 1}$ as indicated in the diagram below:



Notice that Newton's Law of Gravitation automatically satisfies Newton's 3rd Law of Motion since (1) the magnitudes of the forces are equal and (2) they have opposite directions.

The Cavendish Experiment to Measure the Constant of Universal Gravitation G:

How Newton came up equation (1) is somewhat of a mystery but it has been verified in the laboratory involving two masses in the kilogram range. If you calculate the gravitational force between to masses in the kilogram range, you will see that the force is very small. Below is a diagram from Giancoli showing the apparatus first used by Cavendish to verify equation (1) and measure the gravitational constant G. The force F is so small in this experiment that F is measured by the twisting of a thin fiber.

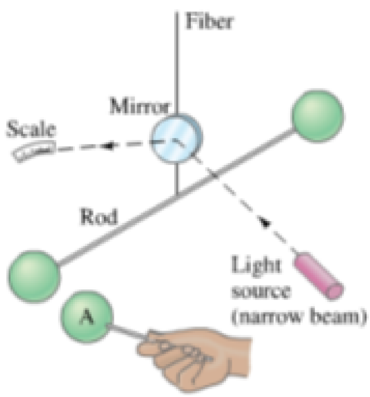


FIGURE 5–20 Schematic diagram of Cavendish's apparatus. Two spheres are attached to a lightweight horizontal rod, which is suspended at its center by a thin fiber. When a third sphere labeled A is brought close to one of the suspended spheres, the gravitational force causes the latter to move, and this twists the fiber slightly. The tiny movement is magnified by the use of a narrow light beam directed at a mirror mounted on the fiber. The beam reflects onto a scale. Previous determination of how large a force will twist the fiber a given amount then allows one to determine the magnitude of the gravitational force between two objects.

NUMERICAL EXAMPLE: The force between two masses M_1=0.5 kg and M_2= 0.2 kg which are 0.3 cm apart is

$$M1 = 0.5\ \mathrm{kg};$$
$$M2 = 0.2\ \mathrm{kg};$$
$$R = 0.3\ \mathrm{cm} * \frac{1\ \mathrm{m}}{100\ \mathrm{cm}};$$
$$G = 6.67 * 10^{-11}\ \frac{\mathrm{Nt}\ \mathrm{m}^2}{\mathrm{kg}^2};$$
$$F = G * \frac{M1 * M2}{R^2}$$

7.41111×10^{-7} Nt

The force of gravity W between a person having mass M=100 kg and the Earth is given by W = M g

where g=9.8 m/s^2 is

```
M = 100. kg;
g = 9.8 m / s^2;
W = M * g
```

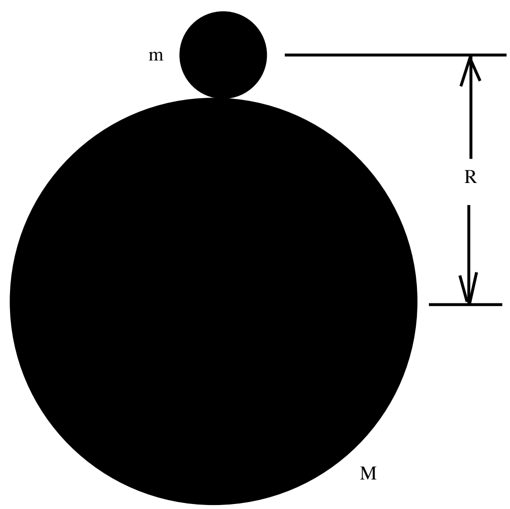
$$\frac{980.\ \text{kg m}}{\text{s}^2}$$

Thus the force of gravity in this case is 980 Nt. This is huge in comparison with the force in the first calculation. The force of gravity W is this large because one of the mass (that if the Earth) is so large.

◀ | ▶

EXAMPLE: Determine the Mass of the Earth

It is possible to measure the mass of the Earth M using the Law of Gravitation. The usual method of putting an object on a balance to determine the mass of the Earth is obviously out of the question. Start by considering a mass m on the surface of the Earth. (The numerical value of m is not important since it cancel out of the calculation below.) We actually now have two ways of calculating the gravitational force acting on the mass m: (1) W = mg and (2) $F = \frac{G\,m\,M}{R^2}$ where M is the mass of the Earth and R is the radius of the Earth. A sketch of the situation is below.



Since F=W in equation (2) we may write mg=$W = \frac{G\,m\,M}{R^2}$ and after canceling the mass m we get

$$g = \frac{G\,M}{R^2} \tag{2}$$

and solving for the mass of the Earth M we get

$$M = \frac{g\,R^2}{G} \tag{3}$$

It is fairly easy to get the radius of the Earth $R=6.4\times10^6$ meters. For example, you could sail around the Earth and measure how far you went to obtain the circumference $C=2\pi R$ and then solve for R. $g=9.8\ m/s^2$ and $G=6.67\times10^{-11}\ \frac{\text{Nt}-m^2}{\text{kg}^2}$ so computing M we get

```
g = 9.8;
G = 6.67 * 10^-11;
R = 6.4 * 10^6;
M = g * R^2 / G
```

6.01811×10^{24}

So the mass of the Earth is 6.0×10^{24} kg from Newton's Law of Gravitation and it should be fairly impressive that M can be determined all.

◄ | ►

EXAMPLE: The value of the acceleration of gravity g on top of Mt. Everest

First consider the force on a mass m at the top of Mt. Everest which has a height h above sea level. Calculating the force on m using W=m$\mathbb{g}$ where $\mathbb{g}$ is the acceleration of gravity at the top of Everest

and equating this to the force on m using the Law of Gravitation yields

$$m\,\mathbb{g} = \frac{G\,m\,M}{(R+h)^2} \tag{4}$$

The mass m cancels out since m appears on both sides of the equations above thus

$$\mathbb{g} = \frac{G\;M}{(R+h)^2} \tag{5}$$

and using equation (3) to eliminate the mass of the Earth yields

$$\mathbb{g} = g\,\frac{R^2}{(R+h)^2} \tag{6}$$

where again R is the radius of the Earth. Looking at equation (6) you can see that since R+h > R the acceleration of gravity on top of Everest is going to be smaller than g=9.8 m/s^2. Recall the radius of the Earth R=6.4×10^6 meters and Everest is h=8848 meter above sea level as obtained at http://en.wikipedia.org/wiki/Mount_Everest.

Everest from Kala Patthar in Nepal

and Everest is located at

Location

Location within Nepal on the Nepal–China border

Location	Solukhumbu District, Sagarmatha Zone, Nepal Tingri County, Xigazê Prefecture, Tibet Autonomous Region, People's Republic of China[2]
Range	Mahalangur Himal, Himalayas
Coordinates	27°59′17″N 86°55′31″E [3]
Climbing	
First ascent	29 May 1953 Edmund Hillary Tenzing Norgay
Easiest route	South Col (Nepal)

But getting back to the acceleration of gravity 𝕘 on top of Everest we get numerically

```
g = 9.8;
h = 8848.;
R = 6.4 * 10^6;
𝕘 = g * R^2/(R + h)^2
```

$$\mathbb{g} = g * \frac{R^2}{(R+h)^2}$$

```
9.77296
```

Well $\mathfrak{g}$=9.77 m/s^2 is smaller than g=9.8 m/s^2 in fact the percent difference is

$$\frac{g - \mathfrak{g}}{g}$$

0.00275928

or about 0.3% which is pretty small but none-the-less measurable.

◀ | ▶

Example: The Acceleration of Gravity on the Moon's Surface

You could take the mass m to the Moon and the mass would remain the same but the force of gravity would be less. The force of gravity would be W=m $\mathcal{g}$ where $\mathcal{g}$ is the acceleration of gravity on the Moon and also the force of gravity acting on the mass m is given by the Law of Gravitation W=F=$\frac{G m \mathcal{M}}{R^2}$ where $\mathcal{M}$ is the mass of the Moon. Since both ways of calculating the gravitational force should yield the same answer if follows

$$m \mathcal{g} = \frac{G m \mathcal{M}}{R^2} \tag{7}$$

and canceling m which appears on both sides we get $\mathcal{g}$ on the Moon

$$\mathcal{g} = \frac{G \mathcal{M}}{\mathcal{R}^2} \tag{8}$$

The mass of the Moon is $\mathcal{M}$=7.35 × 10^{22} kg and the Moon's radius is $\mathcal{R}$=1738000. meters so

$\mathcal{M}$=7.35 × 10^{22} kg

```
ℛ = 1 738 000.;
ℳ = 7.35 * 10^22;
G = 6.67 * 10^-11;
g = G ℳ / ℛ^2
1.62298
```

So the acceleration of gravity on the Moon is $\mathcal{g}$=1.6 m/s^2 which is quite a bit smaller than g=9.8 m/s^2 on Earth.

```
1.6 / 9.8
0.163265
```

So $\mathcal{g}$=0.16 g or on the Moon the acceleration of gravity is about 16% that on Earth.

A Numerical Example: Suppose an object has mass of 60 kg. What is its weight on Earth?

```
m = 60.;
g = 9.8;
W = m g
```

```
588.
```

So on Earth the weight is 588 Nt. What is the weight on the Moon?

```
m = 60.;
g = 1.6;
W = m g
```

```
96.
```

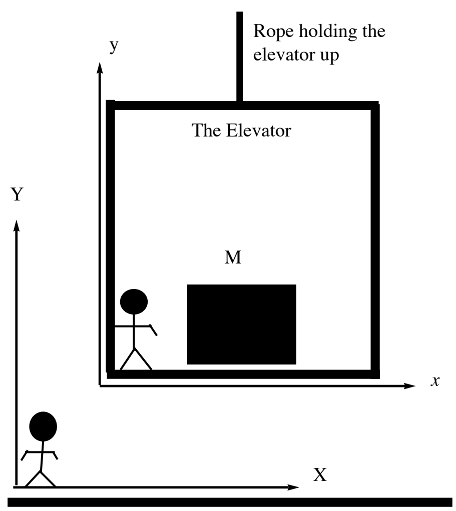
So on the Moon the weight is 96 Nt. which is quite a bit less than 588 Nt.

◄ | ►

Non-Inertial Reference Frames or Coordinate Systems.

Sometimes it is useful to use a coordinate system for which Newton's 1st Law or the law of inertia does NOT hold. These coordinate systems are non-inertial frames of reference. If you use Newton's 2nd law in a non-inertial coordinate system then you must include a correction.

A coordinate system inside a moving ordinary elevator is an example of a non-inertial reference frame. First consider an elevator at rest with respect to the ground and a mass M inside and on the floor of the elevator. Actually there is an ordinary bathroom scale between the mass M and the floor of the elevator so the normal force N can be measured. Later we will take M=5 kg as a numerical example. A sketch of the situation appears



There are two coordinate system one of which is fixed to the ground and is called X, Y. The other coordinate system is fixed to the elevator and is called x, y. The basic idea is compare how an observer fixed to the X, Y coordinate system (on the ground) would view things in comparison to an observer in the x,y coordinate system (on the elevator). The walls of the elevator are clear glass or plastic so the observer X, Y outside can read the bathroom scale even though the scale is inside the elevator. Both observer X, Y and x,y read the same number off of the bathroom scale.

Elevator Problem Continued:

CASE #1: T**he elevator is not moving**. Then an observer on the ground will apply Newton's 2nd law by noting that mass M is not moving so the acceleration A=0 in the Y-direction so the net force in the Y-direction is zero. There are two forces in the Y-direction, the force of gravity -W and the normal force N upward of the floor of the elevator on the mass M. The sum of these forces is zero by Newton's 2nd law so

$$N - W = M\mathbb{a} = 0 \quad \text{since } \mathbb{a} = 0 \tag{9}$$

So N=W and W=Mg. **EXAMPLE:** M= 5kg so the force of gravity is W=5 kg × 9.8 m/s^2 = 49 Nt

The normal force N = 49 Nt and this what the bathroom scale measures.

The observer in x, y in the elevator sees the same thing and agrees with the interpretation of Newton's 2nd law.

CASE #2: **The elevator is moving upward with a constant speed**. According to the observer X, Y the acceleration A=0 since the mass is moving at a constant velocity.

Newton's 2nd Law: N-W=MA=0 and N=M (same as case #1).

According to observer x, y the acceleration a=0 since the mass is at rest.

Newton's 2nd Law: N-W=ma=0 so N=W (same as case #1).

So Case #1 and Case #2 are pretty much the same with regard to Newton's 2nd Law.

CASE #3: T**he elevator has an upward acceleration** $\mathbb{a}$.

Viewpoint of observer X, Y on the ground: The mass M moves upward with an acceleration $\mathbb{a}$.

EXAMPLE: The elevator accelerates upward with $\mathbb{a}$=3. m/s^2

Newton's 2nd law: N - W = M $\mathbb{a}$

W=M g (W is the same force of gravity as before.)

The normal force: N = W + M $\mathbb{a}$

$$N = M g + M\mathbb{a} \tag{10}$$

```
3 * 9.8
29.4
```

N=29.4 Nt. is what the bathroom scale reads. N is also called the apparent weight. N > W.

Numerical Example: M=5 kg, a=3. m/s^2, and g=9.8 m/s^2

W=M g = 49 Nt. N=W+Ma = 49 Nt + 5 kg × 3 m/s^2= 49 Nt + 29.4 Nt = 78.4 Nt

N is what the observer on the ground would read on the bathroom scale as it went by (assumng the walls of the elevator were glass and you could see through them. This is all according the observer X, Y fixed with respect to the ground.

Observer x, y on the elevator: The mass M is NOT moving with respect to elevator so the observer x, y says the acceleration is zero a=0 so Newton's 2nd law indicates the net force in the y-direction is zero. The observer x, y knows two forces act on mass M, the upward normal force and the downward force of gravity -W. So when observer x, y applies Newton's 2nd law what follows is N-W=0 and thus N=W. However, the observer x, y knows the force of gravity is W=Mg since observer x, y made an observation of the scale with the elevator at rest. Unfortunately, the scale reads a normal force N > W. (The example has N=78 Nt. and W=49 Nt.) So Newton's 2nd law does NOT hold for observer x, y. Observer x, y has a non-inertial reference frame since Newton's 2nd law does not hold.

However, Newton's 2nd law can be modified by including a "fictional" force so it works for the observer in the elevator. We know that the observer on the ground has no problem using Newton's 2nd law and got N-W=M a. To make this work for the observer in the elevator, all we do is move the Ma to the other side of the equation and get

$$N - W - M a = 0 \tag{11}$$

for mass M at rest with respect to the elevator. Remember $\mathbb{a}$ is the acceleration of the elevator upward. $- M \mathbb{a}$ is called a "fictional" force since there is no cause of this force. An example of a real force or non-fictious force is gravity of Earth which due to the mass of the Earth. Another example of a real force is the normal force of the scale on the mass M. There are other fictional forces, one example is the centrifugal force and another is the coriolis force since these forces are present only in rotating coordinate systems which are considered accelerated coordinate systems. Example: Suppose mass M is moving with respect to the elevator with an acceleration a in the y direction caused by a rope attached to the top of M and having a tension T. Newton's 2nd Law would read

$$T - W - M \mathbb{a} = M a \quad (12)$$

where $\mathbb{a}$ is the acceleration upward of the elevator with respect to the ground.

CASE #4: T**he elevator has an DOWNWARD acceleration** $\mathbb{a}$.

Observer X, Y on the ground: Mass M moves downward with an acceleration -$\mathbb{a}$.

EXAMPLE $\mathbb{a}$ = 3. m/s^2

Newton's 2nd law: N - W = M (-$\mathbb{a}$)

W=M g (W is the force of gravity and the same as before.

Normal force: N = W - M $\mathbb{a}$ or

$$N = M g - M \mathbb{a} \quad (13)$$

N is what the bathroom scale reads. N is also called the apparent weight. N < W.

Numerical Example: M=5 kg, $\mathbb{a}$=3. m/s^2, and g=9.8 m/s^2

W=M g = 49 Nt.   N=W+M$\mathbb{a}$ = 49 Nt - 5 kg × 3 m/s^2= 49 Nt - 15 Nt = 34 Nt

N is what the observer on the ground would read on the bathroom scale as it went by (assuming the walls of the elevator were glass and you could see through them. This is all according the observer X, Y fixed with respect to the ground.

Observer x, y on the elevator: The mass M is NOT moving with respect to elevator so that observer x,y says the acceleration of the mass M is zero a=0 so Newton's 2nd law indicates the net force in the y-direction is zero. The observer x, y knows two forces act on mass M, the upward normal force and the downward force of gravity -W. So when observer x, y applies Newton's 2nd law what follows is N-W=0 and thus N=W. However, the observer x, y knows the force of gravity is W=Mg since observer x, y made an observation of the scale with the elevator at rest. Unfortunately, the scale reads a normal force N < W. So Newton's 2nd law does NOT hold for observer x, y. Observer x, y has a non-inertial reference frame since Newton's 2nd law does not hold.

Numerical Example: W=49. Nt and N=34 Nt so N<W which does NOT agree with Newton's 2nd Law.

However, Newton's 2nd law can be modified so it works for the observer in the elevator. We know that the observer on the ground has no problem using Newton's 2nd law and got   N-W=M(-$\mathbb{a}$).
To make this work for the observer in the elevator, all we do is move the M$\mathbb{a}$ to the other side of the equation and get

$$N - W + M\mathbb{a} = 0 \qquad (14)$$

So Newton's 2nd law has a different form for the observer in the elevator depending on whether the elevator is accelerated upward or downward. This is strange but true. Another example of a non-inertial reference frame is the one used by an observer inside an accelerated automobile. It is pretty much clear that an accelerated car on it side can be treated much like an elevator accelerated upward.