

12. Transformers, Impedance Matching and Maximum Power Transfer

Introduction

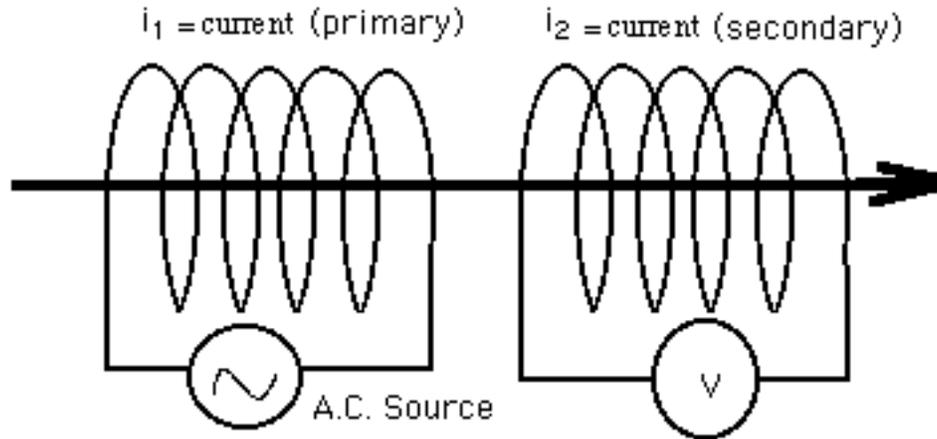
The transformer is a device that takes AC at one voltage and transforms it into another voltage either higher or lower than the original voltage. Alternatively, a transformer can be used to do the same thing with current. Transformers only work with AC (and they do not work with DC) because they work on the principle of Faraday's law of induction which involves time varying magnetic flux. Transformers are useful in electronics because (for example) the voltage supplied by the electric utility is 110-120 voltages while the voltage used by transistors and integrated circuits is typically a much lower (say 10-12 volts).

Additionally transformers can be used to isolate a circuit from the ground in which case the transformer is called an "isolation transformer". Isolation transformer typically have the same number of turns in the primary and secondary coils since there is no need to increase or decrease the voltage when a transformer is used to isolate a circuit.

Finally, transformers can also be used to "impedance match" a power supply with a load to maximize the power transferred to the load. Power supplies have an internal resistance and even something as simple as a battery has an internal resistance which dissipates part of the power generated. This power is wasted since it does not do anything useful. We will show that the maximum power a power supply to transfer to a load resistor is when the load resistance equals the internal resistance of the power supply. Unfortunately, often the load resistor does not equal the internal resistance of the power supply but a transformer can be used to make the effective resistance of the load equal to the power supply and maximum power transfer is achieved.

The Results of Transformers Theory

The basic transformer is two coils wound on an iron core like the diagram below:



One of the coils is connected to the AC source of voltage V_p (or current I_p) and this coil is called the primary coil. Also suppose the primary coil has N_p turns of wire. The load is attached to the secondary coil which we will assume has N_s turns and the voltage across the secondary as measured by a voltmeter is V_s (or an ammeter will measure current I_s). The basic transformer equation is

$$\frac{V_p}{V_s} = \frac{N_p}{N_s} \quad (1)$$

and this is easy to remember as the voltage is proportional to the number of turns. Equation (1) is a result of Faraday's law of induction and this is the reason transformers work only with AC. A transformer is called a "voltage step-up transformer" if the secondary voltage V_s is greater than the primary voltage V_p . For $V_s > V_p$ the ratio of the turns is from equation (1) tells us that $N_s > N_p$. Similarly, a transformer is called a "voltage step-down transformer" if the secondary voltage V_s is less than the primary voltage V_p . For $V_s < V_p$ the ratio of the turns is from equation (1) tells us that $N_s < N_p$.

Transformers conserve energy and power (they do not make energy) if they are ideal. Since Power $P=I \times V$ it follows that if a transformer is voltage step-up, then it must also be current step down. (Or if the transformer is voltage step-down, it is current step-up.) The equation for current corresponding to equation (1) is

$$\frac{I_p}{I_s} = \frac{N_s}{N_p} \quad (2)$$

So the current is inversely proportional to the number of turns.

If we are lucky the manufacturer of a transformer publishes the ratio of turns for the transformer $r = N_p / N_s$. For example, you might be given that $r=2$ or the primary has twice the turns as the secondary. Using this with equation (1) you get

$$\frac{V_p}{V_s} = r \quad (3)$$

and thus

$$V_s = \frac{V_p}{r} \quad (4)$$

So for example, if $V_p=120$ volts the equation (4) tells us that $V_s=120/2=60$ volts.

If we are unlucky and we do not have r the ratio of the turns, we can measure the resistance of the primary R_p and the resistance of the secondary R_s and assuming the length of the wire is proportional to the resistance and that the primary and secondary are the same kind of wire (have the same cross sectional area) we can conclude

$$r = \frac{R_p}{R_s} \quad (5)$$

The calculation of equation (3) and (4) follows as before.

Mostly the transformers we use have the same kind of wire for the primary and secondary. In fact, the only thing that distinguishes the primary from the secondary of the transformer is which coil is connected to the voltage source (this will be the primary) and which coil is connected to the load (and this is called the secondary). For example, the source voltage and load can be reversed in the above transformer so that the ratio of turns $r=1/2$ (instead of $r=2$). Then using equation (3) and (4) with for example, if $V_p=120$ volts, the equation (4) tells us that $V_s = \frac{120}{0.5} = 120 \times 2 = 240$ volts.

Either or both the primary and secondary may be "center tapped" which means the primary coil has a wire running out of the center of the coil. This might be used in constructing certain power supplies like a full wave power supply.

Laboratory Exercises

PART A: You will be supplied with a transformer and using the resistance argument above together with equation (5), measure the ratio of turns r . Attach your SIGNAL GENERATOR in sine wave mode to the primary and set the amplitude say at 12 volts as measured on your oscilloscope. Attach a load resistor something like $R=5,000$ ohms to the secondary and measure the voltage across the load. Use equation (4) to calculate the secondary voltage. Your theoretical prediction should agree pretty much with the experiment.

PART B: Reverse the voltage source and load as attached to the transformer of PART A. Measure the primary voltage and secondary voltage and see if the ratio agrees with your prediction from equation (4) and equation (5).

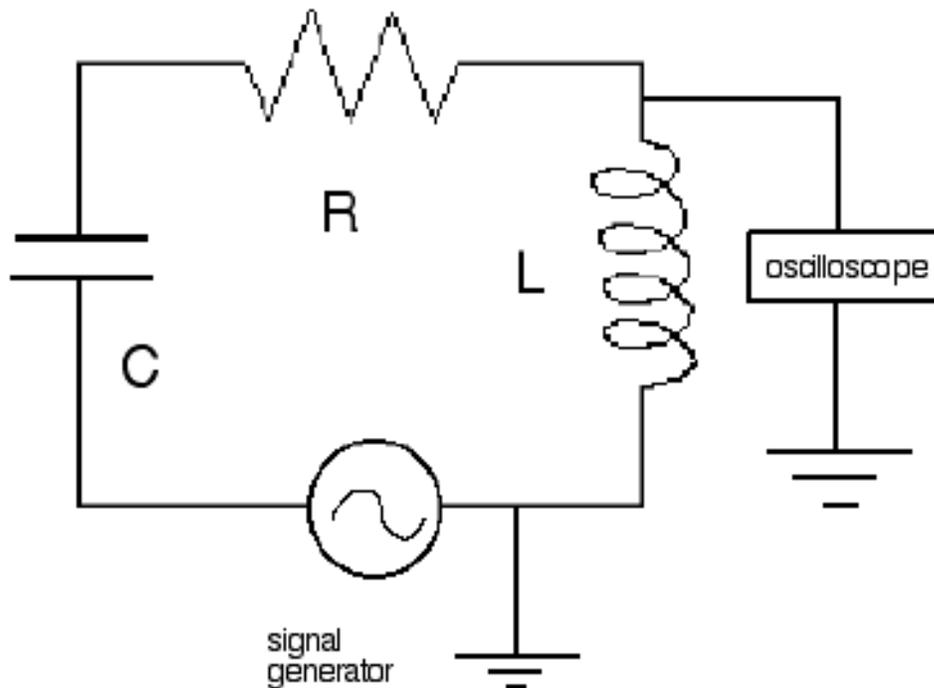
PART C: One of the coils of your transformer is probably center-tapped. Measure the resistance from one end of the coil to the center and see if this is the same as the resistance from the other end to the center. If the two resistances are the same it is safe to assume the coil is divided into two equal parts having the same number of turns.

Use the center tapped coil as the secondary and use only half of the coil (that use the part of the coil from one end to the center). Calculate the ratio of turns r using equation (5). Repeat PART A above with the only difference being that you are using only half of the secondary coil.

PART D: Try to repeat PART A using the 12 volt DC power supply on your circuit box. What happens?

The Use of a Transformer to Isolate a Circuit from Ground

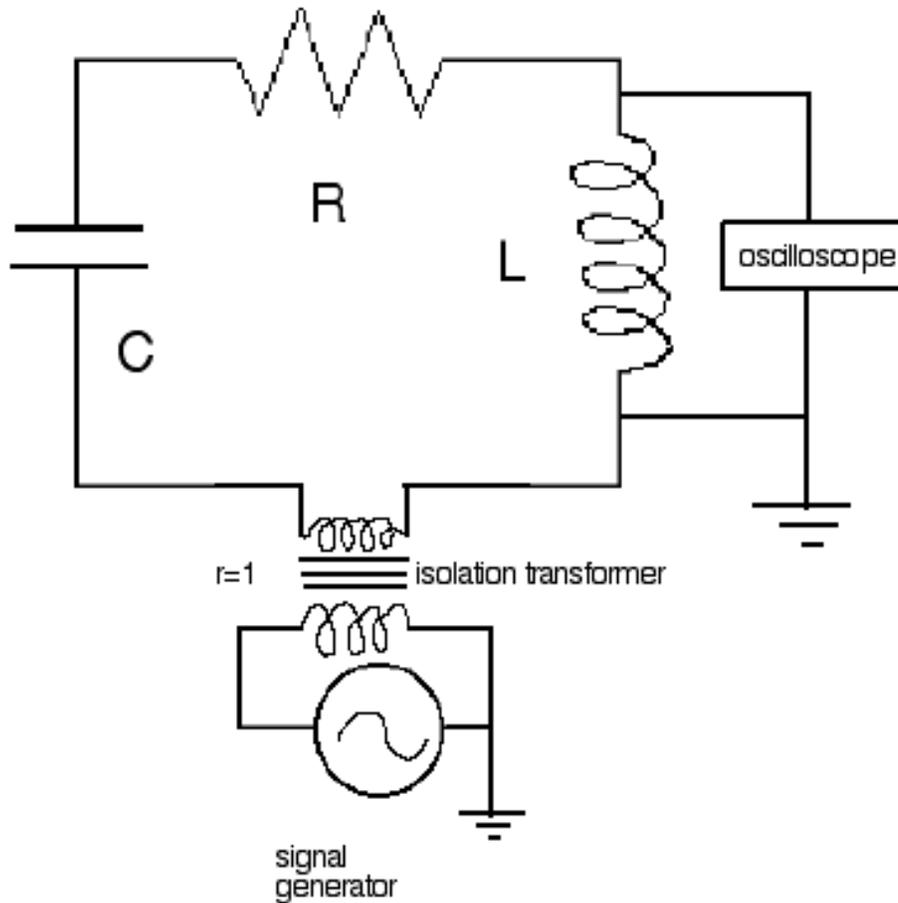
Consider a RLC circuit indicated below:



The signal generator output has a ground as indicated above. This also grounds one side of the inductor L inadvertently. If you use an oscilloscope to measure the voltage across the inductor L , you must attach the center of the probe of the oscilloscope to the top of the inductor L since the outer conductor of the coaxial cable is automatically grounded. Therefore with only one wire attached from the oscilloscope, you measure the voltage across the inductor because the other wire need in the circuit is the ground. This arrangement is unsatisfactory since if you want to measure the voltage across the resistor R , you must interchange the resistor R and inductor L in the above circuit and then attach the oscilloscope probe to the top of the resistor. If you do not interchange the resistor R and inductor L if you attach the

oscilloscope probe to the other side of the resistor, the voltage measured by the oscilloscope actually is the sum of the voltage across the resistor and the voltage across the inductor.

The difficulty with the above circuit can be removed by using an isolation transformer as indicated by the circuit below:



The ground of the oscilloscope and the ground of the signal generator are as indicated. Notice that until the outer coaxial cable of the probe of the oscilloscope is attached as indicated, the oscilloscope will NOT measure any voltage. Without the ground of the coaxial cable attached, there is not a complete circuit which is necessary to measure the voltage across the inductor. The isolation transformer isolates the ground of the signal generator from the ground of the oscilloscope.

If you want to measure the voltage across the resistor, you now do NOT need to interchange the resistor R and inductor L in the above circuit. You need simply attach the oscilloscope leads across the resistor.

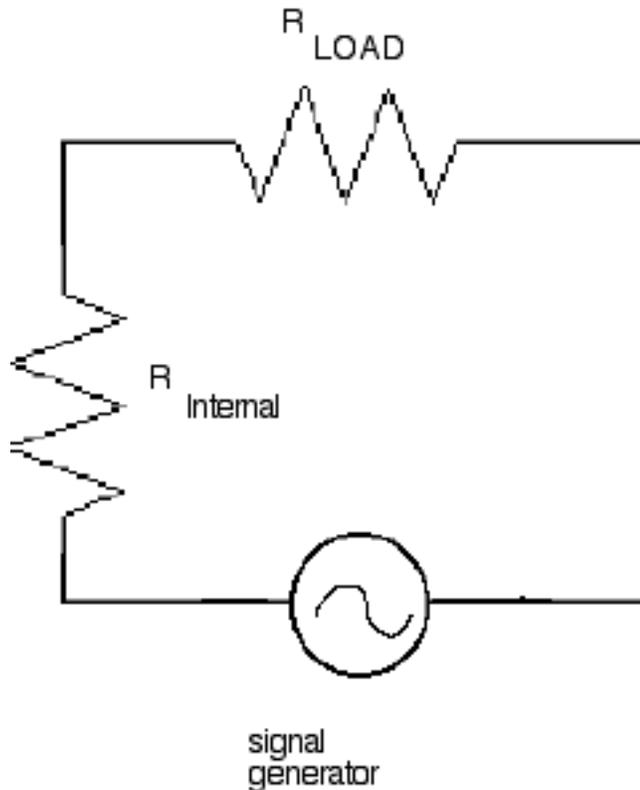
Laboratory Exercise

Build the above RLC circuit with the isolation transformer. Use the values of R, L, and C you used in Lab #11 on Resonance. Show that you can measure the voltage across the inductor L ONLY IF you attach

the oscilloscope leads across the inductor L. Show that you can measure the voltage across the resistor R by attaching the oscilloscope leads across the resistor R.

The Maximum Power Transfer from a Power Supply to a Load Resistor

Consider a power supply connected to a load resistor as indicated by the diagram below:



The signal generator has an internal resistance as indicated and this is an intrinsic part of the power supply. The current in the above circuit is using Ohm's law

$$I = \frac{V}{(R_{\text{Load}} + R_{\text{internal}})} \quad (6)$$

The power transferred to the load resistor from the power supply is

$$P_{\text{Load}} = I^2 R_{\text{Load}} \quad (7)$$

and the power dissipated by the internal resistance of the power supply is

$$P_{\text{internal}} = I^2 R_{\text{internal}} \quad (8)$$

Imagine you can adjust the load resistor until you get the maximum power transferred to the load. That is P_L is a function of R_L and we adjust R_L until P_L is a maximum. Calculus can be used to search for the

value of R_L that makes P_L by taking the derivative

$$\frac{dP_L}{dR_L} = 0 \quad (9)$$

and solving for R_L . Combine equation (6) in equation (7) and obtain

$$P_L = \left(\frac{V}{R_L + R_I} \right)^2 R_L \quad (10)$$

Using *Mathematica* to compute the derivative yields (calling R_L simply R and R_I we call RI)

$$\partial_R \left(\frac{V^2 * R}{(R + RI)^2} \right)$$

$$- \frac{2 R V^2}{(R + RI)^3} + \frac{V^2}{(R + RI)^2}$$

Simplify[%]

$$\frac{(-R + RI) V^2}{(R + RI)^3}$$

The solving for R yields

$$\text{Solve} \left[\frac{(-R + RI) V^2}{(R + RI)^3} == 0, \{R\} \right]$$

{{R → RI}}

The solution to the equation is $R=RI$. What this says is that the power transferred to the load P_L is a maximum when the load resistance R (or what we previously called R_L) is equal to RI (what we previously called R_I the internal resistance). When $R=RI$ half the power of the supply is transferred to the load resistor and but half the power is lost to the internal resistance. The maximum efficiency achieved by a power supply is 50% which is kind of disappointing but true.

$$R_{int} = R_L \quad \text{for maximum power transfer} \quad (11)$$

Laboratory Exercise

Attach a load resistor R_{Load} as indicated in the circuit above and measure the voltage of the signal generator and the voltage across the resistor using an oscilloscope. Start with a load resistor in the neighborhood of $R_{Load} = 5,000 \Omega$. Calculate the power transferred to the load resistor using

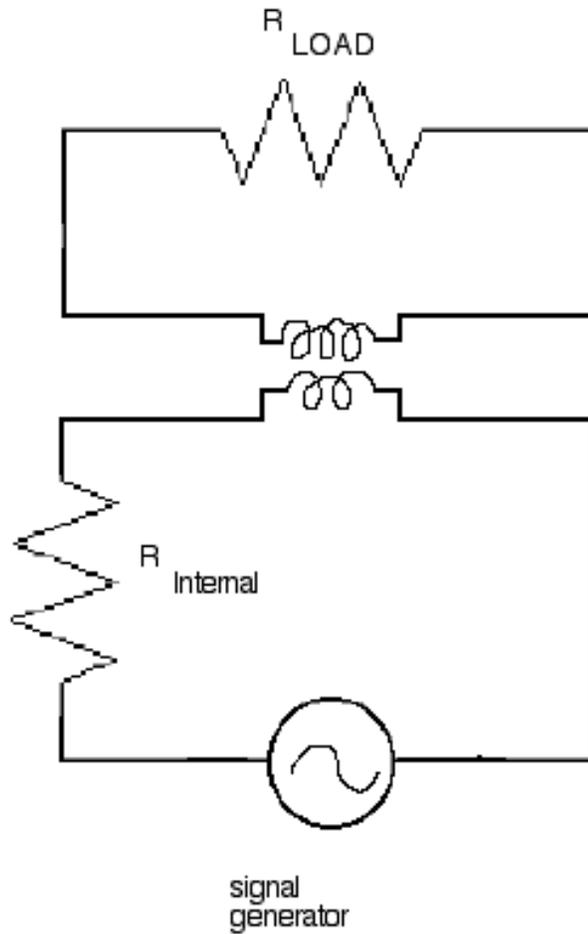
$$P_L = V_L^2 / R_L.$$

Next pick a new load resistor say for example $R_{\text{Load}} = 10,000 \Omega$ and measure the voltage across the load resistor and calculate the power transferred to the load resistor using $P_L = V_L^2 / R_L$. Make sure the voltage provided by the signal generator is the same as before by attaching the oscilloscope across the signal generator. If the voltage of the signal generator has changed, adjust the output level of the signal generator.

Repeat the above for increasing values of R_L using $20,000\Omega$, $40,000\Omega$, $100,000 \Omega$ and $1,000,000 \Omega$ load resistor. Compute the power transferred to the load resistor in each case. Graph P_L versus R_L . Your graph should have a peak and that peak will occur when $R_L = R_{\text{int}}$. This is a method of determining the internal resistance of a power supply.

The Effective Resistance of a Load Resistance attached to a Transformer

Consider a transformer connected between the load resistor and the power supply as indicated below:



Assume the transformer is ideal so that the resistance of the windings is small compared with the load resistor and internal resistance of the power supply. Suppose an ammeter is placed in the secondary circuit and I_s is measured and a voltmeter (or oscilloscope since you are measuring AC) placed across the secondary reads V_s . Ohm's law holds for the secondary circuit so

$$V_s = I_s R_{\text{Load}} \quad (12)$$

A voltmeter (oscilloscope) attached across the primary coil would read V_p and this is different from the voltage of the signal generator V_0 because there is a voltage drop $I_p R_I$ across the internal resistance.

We can write

$$V_0 = I_p R_I + V_p \quad (13)$$

The voltage of the primary V_p is related to the voltage of the secondary V_s via equation (1) so

$$V_p = \frac{N_p}{N_s} V_s \quad (14)$$

and using equation (11) for the voltage of the secondary V_s in equation (13) yields

$$V_p = \frac{N_p}{N_s} I_s R_{Load} \quad (15)$$

However, the secondary current I_s in equation (14) is related to the primary current I_p via equation (2) so

$$I_s = \left(\frac{N_p}{N_s} \right) I_p \quad (16)$$

Finally utilizing equation (15) in equation (14) yields

$$V_p = \left(\frac{N_p}{N_s} \right)^2 I_p R_{Load} \quad (17)$$

which just seems like a complicated equation. However, if you use equation (16) to eliminate V_p in equation (12) you obtain

$$V_0 = I_p R_I + \left(\frac{N_p}{N_s} \right)^2 I_p R_{Load} \quad (18)$$

or more simply after factoring I_p from both terms

$$V_0 = I_p \left(R_I + \left(\frac{N_p}{N_s} \right)^2 R_{Load} \right) \quad (19)$$

Equation (18) should be interpreted as Ohm's law for the primary circuit where you have two resistors in series. One resistor is R_I the internal resistance of the power supply (or signal generator). The other series resistor is effectively

$$R_{eff} = \left(\frac{N_p}{N_s} \right)^2 R_{Load} \quad (20)$$

so equation (18) appears simply

$$V_0 = I_p (R_I + R_{eff}) \quad (21)$$

What equation (19) and (20) are saying is that as far as the primary circuit is concerned, the load resistor R_L has an effective resistance given by equation (18). We know from the maximum power transfer theorem that when $R_I = R_{eff}$ the maximum power is transferred from the signal generator to the load resistor. In a practical situation R_{Load} is the resistance of a speaker or head phone or even the resistance of a light bulb and R_{Load} cannot be adjusted for the maximum power transfer since R_{Load} is an intrinsic property of the speaker, head phone, or light bulb. None-the-less maximum power can be transferred by using a transformer with the ratio of turns $r = N_p / N_s$ chosen properly. What this means is that for a given R_L choose a transformer with a ratio r such that R_{eff} calculate from equation (19) namely $R_{eff} = r^2 R_{Load}$ is such that $R_{int} = R_{eff}$. Then maximum power transfer to the load is achieved. This is called impedance

(or resistance) matching since the load resistance is matched to the internal resistance using a transformer to achieve maximum power transfer.

A Numerical Example of Impedance Matching

Suppose your power supply has an internal resistance $R_{\text{int}} = 20 \Omega$ but you want to attach to a load resistor having resistance $R_{\text{load}} = 2,000 \Omega$. (Think of your power supply as an amplifier and the load resistance as headphones attached to your amplifier.) Design a transformer that will enable the maximum power to be transferred to your headphones.

ANSWER: You know from equation (11) that maximum power transfer occurs when $R_{\text{eff}} = R_{\text{int}} = 20 \Omega$. You also know from equation (19) that the effective resistance seen by the power supply is $R_{\text{eff}} = r^2 R_L = r^2 2000 \Omega$. Combining these two equations yields $20 \Omega = R_{\text{eff}} = r^2 2000 \Omega$ so solving for r we obtain $r^2 = 20 \Omega / 2000 \Omega = 1/100$ and $r = 1/10$ is the turns ratio.