PHYSICAL REVIEW A **71**, 050301(R) (2005)

Optimum unambiguous discrimination of two mixed quantum states

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(Received 18 February 2005; published 9 May 2005)

We investigate generalized measurements, based on positive-operator-valued measures, and von Neumann measurements for the unambiguous discrimination of two mixed quantum states that occur with given prior probabilities. In particular, we derive the conditions under which the failure probability of the measurement can reach its absolute lower bound, proportional to the fidelity of the states. The optimum measurement strategy yielding the fidelity bound of the failure probability is explicitly determined for a number of cases. One example involves two density operators of rank d that jointly span a $2d$-dimensional Hilbert space and are related in a special way. We also present an application of the results to the problem of unambiguous quantum state comparison, generalizing the optimum strategy for arbitrary prior probabilities of the states.

DOI: 10.1103/PhysRevA.71.050301 PACS number(s): 03.67.Hk, 03.65.Ta, 42.50.−p

Many applications in quantum communication and quantum cryptography are based on transmitting quantum systems that, with given prior probabilities, are prepared in one from a set of known mutually nonorthogonal states. Since perfect discrimination between nonorthogonal quantum states is impossible, measurement strategies for state discrimination have been developed that are optimized with respect to various criteria [1]. Here we consider unambiguous discrimination, requiring that the outcome of the measurement be error-free. For two mixed quantum states unambiguous discrimination is possible with a finite probability of success if the supports [2] of their density operators are not identical. When the measurement fails, it returns an inconclusive answer but never an error. In the optimal measurement strategy the failure probability is minimum.

The problem of unambiguously discriminating mixed quantum states arises, for instance, when given pure states undergo a specified decoherence process during transmission through a quantum channel, or when the quantum system is known to be in a pure state that has to be assigned to a particular set out of a number of given sets of pure states, with each set corresponding to a mixed state. While for two pure states the minimum failure probability has long since been known [3,4], the study of unambiguous discrimination among mixed states, or sets of pure states, respectively, started only recently [5–11]. A complete solution, determining the minimum achievable failure probability for arbitrary prior probabilities of the states, has been obtained for the special cases of discriminating a pure and a mixed state [5,6], and of two mixed states of rank d in a $(d+1)$-dimensional joint Hilbert space [7]. For discriminating two arbitrary mixed states, bounds have been derived for the failure probability [7], in terms of the fidelity of the states. In this paper we perform a more detailed analysis, investigating the conditions under which the lowest bound, proportional to the fidelity, can be reached, and deriving also the von Neumann measurements for unambiguous discrimination.

We start by recalling that a measurement for distinguishing two quantum states, characterized by the density operators ρ_1 and ρ_2 and the prior probabilities η_1 and $\eta_2=1-\eta_1$, respectively, can be formally described by three positive operators Π_k with $\Sigma_{k=0}^{2}\Pi_k=I$, where I is the identity. These detection operators are defined in such a way that $\mathrm{Tr}(\rho\Pi_k)$ with $k=1$, 2 is the probability that a system prepared in a state ρ is inferred to be in the state ρ_k, while $\mathrm{Tr}(\rho\Pi_0)$ is the probability that the measurement fails to give a definite answer. When all detection operators are projectors, the measurement is a von Neumann measurement, otherwise it is a generalized measurement based on a positive operator-valued measure (POVM). From the detection operators Π_k schemes for realizing the measurement can be obtained [12,13].

It is our aim to investigate the optimum measurement strategy that minimizes the total failure probability

$$Q=\eta_1\mathrm{Tr}(\rho_1\Pi_0)+\eta_2\mathrm{Tr}(\rho_2\Pi_0). \tag{1}$$

From the relation between the arithmetic and the geometric mean and from the Cauchy-Schwarz inequality [10,14], it follows that $Q\geqslant 2\sqrt{\eta_1\eta_2\mathrm{Tr}(\rho_1\Pi_0)\mathrm{Tr}(\rho_2\Pi_0)}\geqslant 2\sqrt{\eta_1\eta_2}\mathrm{Max}_U|\mathrm{Tr}(U\sqrt{\rho_1}\Pi_0\sqrt{\rho_2})|$, where U describes an arbitrary unitary transformation. The failure probability takes its absolute minimum when the two equality signs hold. This is true if and only if both the relations $\eta_1\mathrm{Tr}(\rho_1\Pi_0)=\eta_2\mathrm{Tr}(\rho_2\Pi_0)$ and $U\sqrt{\rho_1}\sqrt{\Pi_0}\sim\sqrt{\rho_2}\sqrt{\Pi_0}$ are fulfilled. From the first relation we conclude that the number of inconclusive results is equally distributed among the two incoming states. After multiplying the second relation with its Hermitian conjugate, the two conditions for equality can be combined to yield $\sqrt{\Pi_0}(\eta_2\rho_2-\eta_1\rho_1)\sqrt{\Pi_0}=0$. Since in the POVM formalism the detection operators transform a quantum state according to $\rho\rightarrow\Sigma_k\sqrt{\Pi_k}\rho\sqrt{\Pi_k}$ [13], it follows that the total failure probability is smallest when in case of failure the two density operators are transformed into states that are identical after normalization and therefore cannot be further discriminated.

We now recall that unambiguous discrimination of two states leads to the requirement $\rho_1\Pi_2=\rho_2\Pi_1=0$ [1]. Substituting $\Pi_0=I-\Pi_1-\Pi_2$ into the inequality for the failure probability Q [10], given above, we arrive at

$$Q \geqslant 2\sqrt{\eta_1\eta_2}\mathrm{Max}_U|\mathrm{Tr}(U\sqrt{\rho_1}\sqrt{\rho_2})| = 2\sqrt{\eta_1\eta_2}F, \qquad (2)$$

where $F=\mathrm{Tr}[(\sqrt{\rho_2}\rho_1\sqrt{\rho_2})^{1/2}]$ is the fidelity [14]. Using a different method, it has been found already previously by Rudolph *et al.* [7] that

$$Q \geqslant \begin{cases} 2\sqrt{\eta_1\eta_2}F = Q_0 & \text{if } F \leqslant \sqrt{\dfrac{\eta_1}{\eta_2}} \leqslant \dfrac{1}{F} \\ \eta_{\min} + \eta_{\max}F^2 & \text{otherwise}, \end{cases} \qquad (3)$$

with $\eta_{\min}(\eta_{\max})$ denoting the smaller (larger) of the prior probabilities. Here, in addition, we obtained the necessary and sufficient conditions that the detection operators have to fulfill in order to reach the fidelity bound Q_0. They can be summarized as

$$\Pi_0 = I - \Pi_1 - \Pi_2 \geqslant 0, \quad \Pi_1 \geqslant 0, \quad \Pi_2 \geqslant 0, \qquad (4)$$

$$\rho_1\Pi_2 = \rho_2\Pi_1 = 0, \qquad (5)$$

$$\eta_1\mathrm{Tr}(\rho_1\Pi_0) = \eta_1[1 - \mathrm{Tr}(\rho_1\Pi_1)] = \sqrt{\eta_1\eta_2}F, \qquad (6)$$

$$\eta_2\mathrm{Tr}(\rho_2\Pi_0) = \eta_2[1 - \mathrm{Tr}(\rho_2\Pi_2)] = \sqrt{\eta_1\eta_2}F. \qquad (7)$$

In the following we investigate the conditions under which detection operators exist that satisfy Eqs. (4)–(7). For this purpose we use the spectral representations

$$\rho_1 = \sum_{l=1}^{d_1} r_l|r_l\rangle\langle r_l|, \quad \rho_2 = \sum_{m=1}^{d_2} s_m|s_m\rangle\langle s_m|, \qquad (8)$$

where $r_l, s_m \neq 0$, and $\langle r_l|r_m\rangle = \delta_{l,m} = \langle s_l|s_m\rangle$. Furthermore, we introduce the projection operators

$$P_1 = \sum_{l=1}^{d_1} |r_l\rangle\langle r_l|, \quad P_2 = \sum_{m=1}^{d_2} |s_m\rangle\langle s_m|, \qquad (9)$$

and the non-normalized states $|r_l^{\|}\rangle = P_2|r_l\rangle$. We can construct a complete orthonormal basis $\{|h_k\rangle\}$ in the subspace $\mathcal{H}_{1\|}$ spanned by the state vectors $P_2|r_l\rangle$, using the recursion relation $|\tilde{h}_k\rangle = P_2|r_k\rangle - \sum_{i=1}^{k-1}|h_i\rangle\langle h_i|P_2|r_k\rangle$ and determining $|h_k\rangle = |\tilde{h}_k\rangle/\|\tilde{h}_k\|$ [14]. The dimensionality $d_{1\|}$ of $\mathcal{H}_{1\|}$ is equal to the rank of the matrix formed by the elements $\langle r_l|P_2|r_n\rangle$. Similarly, in the subspace $\mathcal{H}_{1\perp}$ that is spanned by the non-normalized vectors $|r_l^{\perp}\rangle = (I-P_2)|r_l\rangle$, we can obtain an orthonormal basis $\{|v_i\rangle\}$ of dimension $d_{1\perp}$. The respective projection operators into the two orthogonal subspaces are

$$P_{1\|} = \sum_{k=1}^{d_{1\|}} |h_k\rangle\langle h_k|, \quad P_{1\perp} = \sum_{i=1}^{d_{1\perp}} |v_i\rangle\langle v_i|, \qquad (10)$$

where $\rho_2|v_i\rangle = 0$. The operator $P_{10} = P_{1\|} + P_{1\perp}$ projects onto a subspace $\mathcal{H}_{10}$ of dimension $d_{1\|} + d_{1\perp}$. Noticing that $\mathrm{Tr}[(P_{10} - P_1)\rho_1] = 0$, we construct the operator

$$\overline{P_1} = P_{1\|} + P_{1\perp} - P_1 = \sum_{j=0}^{\overline{d_1}} |\overline{r_j}\rangle\langle\overline{r_j}|, \qquad (11)$$

where $\rho_1|\overline{r_j}\rangle = 0$. The states $\{|\overline{r_j}\rangle\}$ form an orthonormal basis in the $\overline{d_1}$-dimensional subspace of $\mathcal{H}_{10}$ that is spanned by all states that are orthogonal to P_1, where $\overline{d_1} = d_{1\|} + d_{1\perp} - d_1$. The identity is then given by

$$I = P_{1\perp} + P_2 = P_{1\perp} + P_{1\|} + P_2' = P_1 + \overline{P_1} + P_2'. \qquad (12)$$

Here the operator $P_2' = I - P_{1\perp} - P_{1\|}$ projects onto the subspace $\mathcal{H}_2'$ spanned by those states that are orthogonal to both $P_{1\perp}$ and $P_{1\|}$, implying that $\rho_1 P_2' = 0$. Instead of decomposing the eigenstates of ρ_1, we might as well have started from $|s_m\rangle = P_1|s_m\rangle + |s_m^{\perp}\rangle$, obtaining instead of Eq. (12) the alternative decomposition

$$I = P_{2\perp} + P_1 = P_{2\perp} + P_2 + P_1' = P_2 + \overline{P_2} + P_1', \qquad (13)$$

where the projectors are defined analogously.

Now we can specify the general structure of all detection operators, Π_1 and Π_2, that describe unambiguous discrimination, i.e., satisfy Eqs. (4) and (5). We write

$$\Pi_1 = \sum_{j=1}^{d_{1\perp}} \alpha_j'|v_j'\rangle\langle v_j'| = \sum_{i,j=1}^{d_{1\perp}} \alpha_{ij}|v_i\rangle\langle v_j|, \qquad (14)$$

where $0 \leqslant \alpha_j' \leqslant 1$ and $|v_j'\rangle = \Sigma_i u_{ji}|v_i\rangle$ with $\{u_{ji}\}$ being a unitary matrix. We note that $\Sigma_j|v_j'\rangle\langle v_j'| = P_{1\perp}$ since the eigenstates $|v_j'\rangle$ form a complete orthonormal basis in $\mathcal{H}_{1\perp}$. For representing Π_2 we start from the same decomposition of the identity, and take into account that none of the eigenstates of Π_0 must be contained in the subspace $\mathcal{H}_2'$ when the failure probability is to be as small as possible. This leads to

$$\Pi_2 = \sum_{i=1}^{\overline{d_1}} \beta_i'|\overline{r_i'}\rangle\langle\overline{r_i'}| + P_2' = \sum_{i,j=1}^{\overline{d_1}} \beta_{ij}|\overline{r_i}\rangle\langle\overline{r_j}| + I - P_{10}, \qquad (15)$$

where $0 \leqslant \beta_i' \leqslant 1$ and $\Sigma_{i=1}^{\overline{d_1}}|\overline{r_i'}\rangle\langle\overline{r_i'}| = \overline{P_1}$. The constants α_{ij} and β_{ij} are subject to the constraint that $\Pi_0 \geqslant 0$.

Clearly, when $P_1 = P_{1\|} = I$, and consequently also $P_2 = P_{2\|} = I$, it follows that $\Pi_1 = \Pi_2 = 0$ and $\Pi_0 = I$, yielding a unit failure probability that makes error-free discrimination impossible. We therefore require that $P_{1\perp} \neq 0$, or $P_{2\perp} \neq 0$, respectively, which, because of normalization, is equivalent to

$$\mathrm{Tr}(P_1\rho_2) < \mathrm{Tr}(P_1\|\rho_2, \quad \mathrm{Tr}(P_2\rho_1) < \mathrm{Tr}(P_2\|\rho_1. \qquad (16)$$

Before studying the optimum measurement, let us consider the von Neumann measurements for unambiguous discrimination. If $\alpha_j' = 0$ for all j, and $\beta_i' = 1$ for all i, it follows that $\Pi_1 = 0$ and $\Pi_2 = \overline{P_1} + P_2'$. Hence $\Pi_0 = P_1$, with the failure probability $Q_{N1} = \eta_1 + \eta_2\mathrm{Tr}(P_1\rho_2)$. Another von Neumann measurement is generated when $\alpha_j' = 1$ for all j, and $\beta_i' = 0$ for all i, giving $\Pi_1 = P_{1\perp}$ and $\Pi_2 = P_2'$. Then $\Pi_0 = P_{1\|}$, with the failure probability

$$Q_{N1} = \eta_1 \mathrm{Tr}(P_2\rho_1) + \eta_2 \mathrm{Tr}(P_{1\|}\rho_2), \tag{17}$$

where the relation $\mathrm{Tr}(P_{1\|}\rho_1)=1-\mathrm{Tr}(P_{1\perp}\rho_1)=\mathrm{Tr}(P_2\rho_1)$ has been applied. In this measurement the state is unambiguously found to be ρ_1 when a detector click occurs in a direction orthogonal to all eigenstates of ρ_2. On the other hand, for a click in a direction orthogonal to both $P_{1\|}$ and $P_{1\perp}$, the state is determined to be ρ_2 with certainty, and in the rest of cases the result is inconclusive. So far we relied on Eq. (12). Based on the complementary decomposition of the identity, Eq. (13), we obtain an alternative pair of von Neumann measurements. These yield the failure probabilities $Q_{N2}=\eta_2+\eta_1\mathrm{Tr}(P_2\rho_1)$ and

$$Q_{N2\|} = \eta_2 \mathrm{Tr}(P_1\rho_2) + \eta_1 \mathrm{Tr}(P_{2\|}\rho_1). \tag{18}$$

Obviously $Q_{N2\|}\leqslant Q_{N1}$ and $Q_{N1\|}\leqslant Q_{N2}$.

We now return to the optimum measurement. Since the von Neumann measurements can be performed for arbitrary given parameters, the optimized failure probability certainly obeys the inequality

$$Q_{\mathrm{opt}} \leqslant \mathrm{Min}\{Q_{N1\|}, Q_{N2\|}\}. \tag{19}$$

According to Eqs. (6) and (7) the absolute minimum of the failure probability, $Q_0=2\sqrt{\eta_1\eta_2}F$, is reached if and only if the two conditions $\mathrm{Tr}(\rho_1\Pi_0)/F=\sqrt{\eta_2/\eta_1}$ and $F/\mathrm{Tr}(\rho_2\Pi_0)=\sqrt{\eta_2/\eta_1}$ are fulfilled. However, due to the structure of the operators Π_1 and Π_2, the possible values of $\mathrm{Tr}(\rho_k\Pi_0)=1-\mathrm{Tr}(\rho_k\Pi_k)$, for $k=1$, 2, have a lower bound. In particular,

$$\mathrm{Tr}(\rho_1\Pi_0) \geqslant 1 - \mathrm{Tr}(P_{1\perp}\rho_1) = \mathrm{Tr}(P_2\rho_1), \tag{20}$$

$$\mathrm{Tr}(\rho_2\Pi_0) \geqslant \mathrm{Tr}(P_{1\|}\rho_2) - \mathrm{Tr}(\bar{P}_1\rho_2) = \mathrm{Tr}(P_1\rho_2), \tag{21}$$

where in the first equation the equality sign holds when $\alpha'_j=1$ in Eq. (14), and in the second equation the equality is reached when $\beta'_i=1$ in Eq. (15). Therefore we obtain that the *condition*,

$$\frac{\mathrm{Tr}(P_2\rho_1)}{F} \leqslant \sqrt{\frac{\eta_2}{\eta_1}} \leqslant \frac{F}{\mathrm{Tr}(P_1\rho_2)}, \tag{22}$$

is *necessary*, i.e., the fidelity bound, $Q=Q_0$, can only be reached in part or in the whole of this interval.

The interval specified by Eq. (22) is not empty only when $\mathrm{Tr}(P_2\rho_1)\mathrm{Tr}(P_1\rho_2)\leqslant F^2$. For two density operators that violate this inequality, the failure probability Q_0 cannot be achieved for any values of the prior probabilities of the states, and the conditions (6) and (7) are then of no help for determining the optimum measurement. Moreover, our result shows that in general the lower bound Q_0 can only be reached in an interval of the ratio η_2/η_1 that is smaller than the interval given in Eq. (3), since $\mathrm{Tr}(P_2\rho_1)/F\geqslant F$ and $F/\mathrm{Tr}(P_1\rho_2)\leqslant 1/F$. The latter relations follow from the general inequalities

$$\mathrm{Tr}(P_2\rho_1)\mathrm{Tr}(P_1\|\rho_2) \geqslant F^2, \quad \mathrm{Tr}(P_1\rho_2)\mathrm{Tr}(P_2\|\rho_1) \geqslant F^2 \tag{23}$$

that can be readily inferred from Eqs. (2), (17), and (18).

The parameter intervals in Eqs. (3) and (22) coincide when $\mathrm{Tr}(P_1\rho_2)=\mathrm{Tr}(P_2\rho_1)=F^2$. This condition is fulfilled, e.g., for density operators of the form $\rho_1=\Sigma_{i=1}^d r_i|r_i\rangle\langle r_i|$ and $\rho_2=\Sigma_{i=1}^d r_i|s_i\rangle\langle s_i|$, with $\langle r_i|s_j\rangle=b\,\delta_{ij}$, where the corresponding eigenvalues are identical. The fidelity is then found to be $F=|b|$.

Another simplification arises when $P_{1\|}$ and $P_{2\|}$ are one-dimensional projectors, $d_{1\|}=d_{2\|}=1$. In this case equality holds in Eqs. (23) [15], which implies that $F^2=\mathrm{Tr}(P_2\rho_1)\mathrm{Tr}(P_{1\|}\rho_2)\geqslant\mathrm{Tr}(P_2\rho_1)\mathrm{Tr}(P_1\rho_2)$, where Eq. (16) has been taken into account. Hence again for any two density operators the necessary condition (22) is fulfilled for a certain range of the ratio η_2/η_1. At the lower limit of this range, i.e., for $\sqrt{\eta_2/\eta_1}=\mathrm{Tr}(P_1\rho_2)/F$, we can write $2\sqrt{\eta_1\eta_2}F=\eta_1F^2/\mathrm{Tr}(P_1\rho_2)+\eta_2\mathrm{Tr}(P_1\rho_2)=Q_{N1\|}$, and similarly we find that at the upper limit $2\sqrt{\eta_1\eta_2}F=Q_{N2\|}$. Thus, if $Q=Q_0$ in the entire range in Eq. (22), the complete solution for the optimum measurement is known.

In general, in order to find the optimum measurement strategy that yields the failure probability Q_0, we have to determine the parameters α_{ij} and β_{ij} in Eqs. (14) and (15) that satisfy the necessary and sufficient conditions (4)–(7). In the following we apply this method to a number of special cases.

First we consider two density operators of rank d in a $2d$-dimensional joint Hilbert space. In such a case $P'_2=0$ and the identity can be alternatively expressed as $I=P_1+\bar{P}_1$ or $I=P_{1\perp}+P_2$, which means that $P_{1\|}=P_2$, $P_{2\|}=P_1$ and $\bar{P}_1=P_{2\perp}$. We start from Eqs. (8) with $d_1=d_2=d$ and assume that $|s_i\rangle=(|r_i\rangle+|\bar{r}_i\rangle)/\sqrt{2}$, and $|v_i\rangle=(|r_i\rangle-|\bar{r}_i\rangle)/\sqrt{2}$ $(i=1,\ldots,d)$. Then we obtain $F=\Sigma_i\sqrt{r_is_i}/2$ and $\mathrm{Tr}(P_1\rho_2)=\mathrm{Tr}(P_2\rho_1)=1/2$. It is important to note that in general there exist sets of eigenvalues $\{r_i\}$ and $\{s_i\}$ where $F^2<1/4$ and the necessary condition, Eq. (22), cannot be fulfilled. In the following, however, we restrict ourselves to the special case that $r_i=s_i$ for $i=1,\ldots,d$, for which $F=1/\sqrt{2}$. The necessary condition for the lower bound Q_0 to be achievable then reads $1/\sqrt{2}\leqslant\sqrt{\eta_2/\eta_1}\leqslant\sqrt{2}$. Further, we find the solutions $\alpha_{ij}=\alpha\,\delta_{ij}$ and $\beta_{ij}=\beta\,\delta_{ij}(i,j=1,\ldots,d)$, where $\alpha=2-\sqrt{2\eta_2/\eta_1}$ and $\beta=2-\sqrt{2\eta_1/\eta_2}$. Π_0 has two eigenvalues, $\lambda_0=0$ and $\lambda_1=2-\alpha-\beta$, each with a d-fold degeneracy. Thus the optimum Π_0 is always an operator of rank d. Note that $2\sqrt{2}-2\leqslant\lambda_1\leqslant 1$ in the whole interval $F\leqslant\sqrt{\eta_1/\eta_2}\leqslant 1/F$. Hence in this parameter interval the optimum detection operators yielding the lower bound Q_0 are $\Pi_1=\alpha P_{1\perp}$ and $\Pi_2=\beta P_{2\perp}$. At the upper and lower limits of the interval the measurement turns into the von Neumann measurements that give the failure probabilities $Q_{N1\|}=Q_{N2}$ and $Q_{N2\|}=Q_{N1}$, respectively. Since in our example $\mathrm{Tr}(P_2\rho_1)=\mathrm{Tr}(P_1\rho_2)=F^2$, we find that $Q_{N1}=\eta_1+\eta_2F^2$ and $Q_{N2}=\eta_2+\eta_1F^2$. Thus we derived a measurement strategy that yields the equality sign in Eq. (3) for two mixed states.

In our next examples we focus on the case $d_{1\|}=d_{2\|}=1$. First we assume that the density operators given in Eq. (8) have arbitrary ranks d_1 and d_2, and that $\langle r_l|s_m\rangle=a\,\delta_{l,1}\delta_{m,1}$ with $|a|<1$. This yields $F=\sqrt{s_1r_1}|a|$, $\mathrm{Tr}(P_2\rho_1)=F^2/s_1$, $\mathrm{Tr}(P_1\rho_2)=F^2/r_1$ and $d_{1\|}=d_{2\|}=1$. For the parameter range specified in Eq. (22), we obtain the optimum detection operators

$$\Pi_1 = \left(1 - \sqrt{\frac{\eta_2}{\eta_1}\frac{F}{r_1}}\right)\frac{|\tilde{v}_1\rangle\langle\tilde{v}_1|}{(1-|a|^2)^2} + \sum_{l=2}^{d_1}|r_l\rangle\langle r_l|,$$

$$\Pi_2 = \left(1 - \sqrt{\frac{\eta_1}{\eta_2}\frac{F}{s_1}}\right)\frac{|\tilde{\tilde{r}}_1\rangle\langle\tilde{\tilde{r}}_1|}{(1-|a|^2)^2} + \sum_{m=2}^{d_2}|s_m\rangle\langle s_m|,$$

where we introduced $|\tilde{v}_1\rangle = |r_1\rangle - a|s_1\rangle$ and $|\tilde{\tilde{r}}_1\rangle = |s_1\rangle - a^*|r_1\rangle$.

This solution can be applied to the problem of quantum state comparison [16], where two identical quantum objects are each prepared either in the state $|\psi_1\rangle$, or in the state $|\psi_2\rangle$, and where we wish to determine unambiguously whether the states are equal or different. The task amounts to distinguishing the two-particle states $\rho_1 = (1/2)(|\psi_1,\psi_1\rangle\langle\psi_1,\psi_1| + |\psi_2,\psi_2\rangle\langle\psi_2,\psi_2|)$ and $\rho_2 = (1/2)(|\psi_1,\psi_2\rangle\langle\psi_1,\psi_2| + |\psi_2,\psi_1\rangle\langle\psi_2,\psi_1|)$, where $F = |\langle\psi_1|\psi_2\rangle|$. Upon determining the eigenstates, we find that the structure of ρ_1 and ρ_2 corresponds to the one treated in the above special example, with $r_1 = s_1 = (1+F^2)/2$ and $|a| = 2F/(1+F^2)$. The minimum failure probability in unambiguous quantum state comparison follows to be

$$Q_{\rm opt} = \begin{cases} 2\sqrt{\eta_1\eta_2}F & \text{if } \sqrt{\dfrac{\eta_{\min}}{\eta_{\max}}} \geqslant \dfrac{2F}{1+F^2} \\ \eta_{\max}\dfrac{2F^2}{1+F^2} + \eta_{\min}\dfrac{1+F^2}{2} & \text{otherwise}. \end{cases} \quad (24)$$

Here $\eta_{\min}(\eta_{\max})$ is the smaller (larger) of the values $\eta_1 = p_1^2 + p_2^2$ and $\eta_2 = 2p_1p_2$, where p_1 and p_2 are the prior probabilities of the states $|\psi_1\rangle$ and $|\psi_2\rangle$, respectively.

As our final example we mention the problem of discriminating a pure state, $\rho_1 = |r_1\rangle\langle r_1|$, from a mixed state ρ_2, or from a set of pure states, respectively, that has been introduced as quantum state filtering [5,17]. In this case $\mathrm{Tr}(P_1\rho_2) = F^2$ and $\mathrm{Tr}(P_2\rho_1) = \|r_1^{\parallel}\|^2$. In the parameter interval given by Eq. (22) the optimum detection operators take the form

$$\Pi_1 = \left(1 - \sqrt{\frac{\eta_2}{\eta_1}}F\right)\frac{|v_1\rangle\langle v_1|}{1-\|r_1^{\parallel}\|^2},$$

$$\Pi_2 = \left(1 - \sqrt{\frac{\eta_1}{\eta_2}}\frac{\|r_1^{\parallel}\|^2}{F}\right)\frac{|\vec{r}_1\rangle\langle\vec{r}_1|}{1-\|r_1^{\parallel}\|^2} + P_2',$$

and the previous solution for the minimum failure probability in optimum unambiguous quantum state filtering [5,6] is readily regained.

In summary, we performed a detailed analysis of the probabilistic measurement for unambiguous discrimination between two arbitrary mixed quantum states. We derived general analytical relations that depend on five quantities characterizing the mutual relationship of the density operators of the states. These quantities are the expressions $\mathrm{Tr}(P_1\rho_2)$ and $\mathrm{Tr}(P_{1\parallel}\rho_2)$, as well as $\mathrm{Tr}(P_2\rho_1)$ and $\mathrm{Tr}(P_{2\parallel}\rho_1)$ and, most importantly, the fidelity F. We also showed that the method developed in this paper can be used to find complete analytical solutions that describe the optimum measurement for special cases.

We thank Philippe Raynal and Norbert Lütkenhaus for sharing a manuscript [18] prior to publication on research along very similar lines and with partially overlapping conclusions. U. H. gratefully acknowledges discussions with O. Benson and the members of his group. J.B. acknowledges useful discussions with E. Feldman, M. Hillery, N. Lütkenhaus, Ph. Raynal, and Y. Sun, as well as partial support from the Humboldt Foundation and PSC-CUNY.

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