Optical Realization of Optimal Unambiguous Discrimination for Pure and Mixed Quantum States

Masoud Mohseni,[1] Aephraim M. Steinberg,[1] and János A. Bergou[2]

[1]Department of Physics, University of Toronto, 60 St. George Street, Toronto, Ontario, Canada M5S 1A7

[2]Department of Physics and Astronomy, Hunter College of the City University of New York, 695 Park Avenue, New York, New York 10021, USA

(Received 24 December 2003; published 9 November 2004)

Quantum mechanics forbids deterministic discrimination among nonorthogonal states. Nonetheless, the capability to distinguish nonorthogonal states unambiguously is an important primitive in quantum information processing. In this work, we experimentally implement generalized measurements in an optical system and demonstrate the first optimal unambiguous discrimination between three nonorthogonal states, with a success rate of 55%, to be compared with the 25% maximum achievable using projective measurements. Furthermore, we present the first realization of unambiguous discrimination between a pure state and a nonorthogonal mixed state.

DOI: 10.1103/PhysRevLett.93.200403 PACS numbers: 03.65.Ta, 03.67.Hk, 42.50.Xa

Quantum measurement theory poses fundamental limitations on the amount of information that can be obtained about the state of a single quantum system. Specifically, it is impossible to perfectly discriminate between two or more nonorthogonal quantum states. However, by appropriately choosing a set of measurements, *nondeterministic* state discrimination is possible if the system has been prepared in a member of a previously specified set of nonorthogonal states. Quantum state discrimination plays an important role in quantum information and quantum communications [1] and is at the heart of quantum cryptography protocols [2].

Several different strategies have been developed to accomplish state-discrimination tasks. "Minimum-error discrimination" (MD) seeks a "best guess" on every trial, minimizing the rate of incorrect guesses. Helstrom showed [3] that, for MD of two states, the optimal strategy can always be achieved by a (von Neumann) projective measurement. "Unambiguous state-discrimination" (UD), on the other hand, seeks to determine with certainty which state the system was in. This can be done only on some fraction of the trials, the others being termed "inconclusive," and the optimal UD success rate *cannot* always be achieved with projective measurements. For the case of two pure nonorthogonal quantum states with equal *a priori* probability, the maximum probability of success was derived by Ivanovic and Dieks and Peres [4]. Clearly, these two strategies may be regarded as limiting cases of a more general approach with both a finite inconclusive rate and a finite error rate [5]. Of course, in any experimentally realistic situation, even an ideal "unambiguous" discrimination strategy will not be error-free. It is of great importance to understand the theoretical and practical limitations, since state discrimination is part of quantum key distribution protocols, and because the effect of a hypothetical eavesdropper's attack on such cryptosystems requires knowledge of the maximum information she could extract.

As for the experimental state of the art, for two nonorthogonal states UD was qualitatively demonstrated by Huttner *et al.* [6], while MD was demonstrated by Barnett and Riis [7]. Clarke *et al.* demonstrated UD quantitatively for a pair of nonorthogonal states [8]. In a second experiment, using highly symmetric trine and tetrad states (linearly dependent states in two dimensions), they performed optimum MD and also a projective measurement which could indicate that a qubit was not prepared in one out of three or four possibilities [9]. Thus, all of the previous experiments have been limited to nonorthogonal pure states of a qubit in a two-dimensional Hilbert space. In the present work, by constructing a multirail optical interferometer enabling us to perform a large class of generalized measurements, we extend these results to higher-dimensional Hilbert spaces with no restriction on the symmetry of the states and explicitly demonstrate that we can achieve a significantly higher success rate than any projective measurement. We perform the first optimal unambiguous state discrimination for three nonorthogonal states, showing that the experimental success rate may be as much as twice as high as for any von Neumann measurement scheme. We also demonstrate the first optimal quantum state "filtering," discriminating between two *subsets* of a set of three nonorthogonal states. This is equivalent to discrimination between a pure and a mixed state [10,11].

Unambiguous state discrimination between N states has $N + 1$ outcomes: the N possible conclusive results, and the inconclusive result. Since no projective measurement in an N-dimensional Hilbert space can have more than N outcomes, generalized measurements are required. Generalized measurements (or positive-operator valued measures, POVMs) provide the most general means of transforming the state of a quantum system [3,12]. POVMs can be implemented by embedding the system into a larger Hilbert space and unitarily entangling it with the extra degrees of freedom (ancilla) [13].

Postselection (projective measurement) of the ancilla induces an effective nonunitary transformation of the original system. By an appropriate design of the entangling unitary, this effective nonunitary transformation can turn an initially nonorthogonal set of states into a set of orthogonal states with a finite probability of success. The optimum strategy is the one that maximizes the average probability of success for this procedure. The generalization of [4] to more than two states was developed by several groups [14,15].

The problem of distinguishing among two *subsets* of a set of N nonorthogonal quantum states has been termed "quantum state filtering" for the case when one subset contains one state and the other $N-1$ [10]. Unambiguous filtering has been studied by Sun *et al.* for $N=3$ in [16] and for arbitrary N in [17]. Quantum state filtering can also be interpreted as unambiguous discrimination between two mixed states. Let us consider the case of three nonorthogonal states, $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$ with *a priori* probabilities of η_1, η_2, and η_3. Filtering is the optimal strategy that can distinguish the state $|\psi_1\rangle$ from the subset $\{|\psi_2\rangle, |\psi_3\rangle\}$. This is equivalent to discriminating the pure state $|\psi_1\rangle\langle\psi_1|$, with *a priori* probability of η_1, from the mixed state $(\eta_2|\psi_2\rangle\langle\psi_2| + \eta_3|\psi_3\rangle\langle\psi_3|)/(\eta_2+\eta_3)$, with *a priori* probability of $\eta_2+\eta_3$.

Here, we present our experimental data for implementing the optimal POVM both for the case of full UD of and filtering from $N=3$ linearly independent nonorthogonal states. In our experiment the generalized measurement was realized by utilizing linear optical elements and photodetectors, based on the proposal of [18]. This can be accomplished by a single-photon representation of the initial states and output states, a multirail optical network for performing the unitary transformation [19,20], and photodetectors at each output port to carry out the required nonunitary transformation. The $N+1$-dimensional unitary operation can be implemented by an appropriate multipath optical interferometer [18,19]. For the case of three nonorthogonal states, $\{|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle\}$, living in a 3D Hilbert space, an eight-port optical interferometer was constructed to perform transformations in the 4D system + ancilla space. All beam splitters in this interferometer were designed (using a combination of polarizing beam splitters and wave plates) to have variable reflectivity, so that the appropriate interferometer could be implemented for any desired discrimination problem (see Fig. 1).

By using beam splitters to send one photon into some linear superposition of the first three rails, we can generate arbitrary quantum states in this three-dimensional Hilbert space, represented as $|\psi\rangle_{\rm in} = \sum_{j=1}^{3} c_j \hat{a}_j^\dagger |0\rangle$, where $\sum_{j=1}^{3} |c_j|^2 = 1$, and $\hat{a}_j^\dagger$ is the creation operator for the jth optical rail. Note that the fourth rail, which acts as the ancilla, never contains a photon. The interferometer is designed to perform the unitary operation U which optimizes state discrimination. It maps the input field operators $\hat{a}_k^\dagger$ into output field operators as $\hat{a}_k^\dagger = \sum_{j=1}^{4} U_{jk}\hat{a}_{j_{\rm out}}^\dagger$, such that the initial state evolves into $|\psi\rangle_{\rm out} = \sum_{j=1}^{j=4}\sum_{k=1}^{k=3} U_{jk} c_k \hat{a}_{j_{\rm out}}^\dagger |0\rangle$. A photon in mode 4 now indicates an inconclusive result. On the other hand, a photon in mode 1, 2, or 3 unambiguously indicates that the initial state was $|\psi_1\rangle$, $|\psi_2\rangle$, or $|\psi_3\rangle$, respectively.

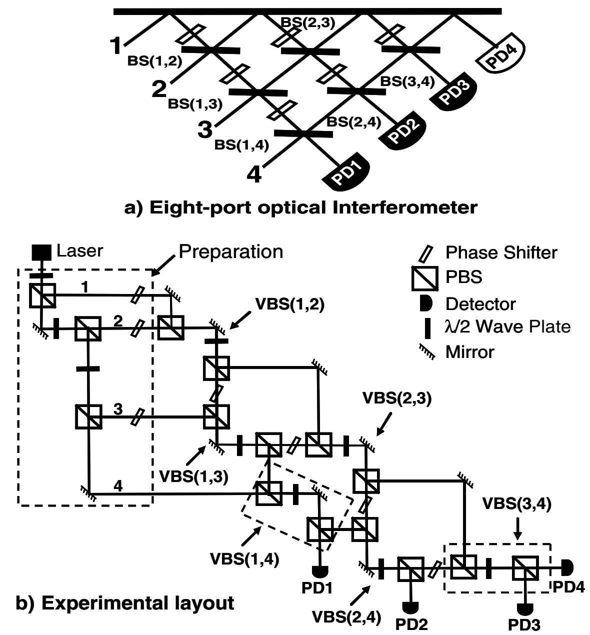

FIG. 1. (a) Eight-port optical interferometer: suitable beam splitters are placed at each crossing of two optical rails to realize any desired unitary transformation on the input states. A detection in rail 4 corresponds to an inconclusive result and a detection in rails 1 to 3 corresponds to states $|\psi_1\rangle$ to $|\psi_3\rangle$. (b) Experimental layout: this interferometer can perform various desired generalized measurement by doing arbitrary unitary operations in four-dimensional Hilbert space and projective measurements at output ports 1 to 4. The variable beam splitters (VBS) realize the corresponding beam splitters in (a) for arbitrary reflectivity and transmissions. Photodiodes PD1 to PD4 detect the photons at the output ports 1 to 4.

The actual experimental setup is shown in Fig. 1(b). To demonstrate unambiguous discrimination, and characterize the success and error rates of our setup, we performed the experiment using a large ensemble of identically prepared photons from a diode laser operating at 780 nm. The nonorthogonal states were prepared by using an arrangement of polarizing beam splitters (PBSs), half-wave plates, and phase shifters. A 1-mm-thick glass slide, at an angle of 55° to the incident beam, was used as a phase shifter. At this angle a differential rotation of 0.05° of the phase shifter produces a π phase shift and causes a beam displacement of less than a micron. Each nonorthogonal state consisted of a different superposition of light in rails 1–3, with the relative field amplitude being adjusted to generate the appropriate coefficients c_j. Rail 4 contained the vacuum for all input states [21]. We designed a VBS that could be placed at each crossing of the beams in Fig. 1(a), in order to perform arbitrary 4D unitaries. The VBS consists of three half-wave plates and two PBSs [22]. The polarizing beam splitters are used to convert information between spatial and polarization degrees of freedom, such that instead of arbitrary coupling between two spatial modes, we need only to implement arbitrary coupling between two polarizations, easily accomplished using wave plates. The setup was designed such that in all interferometers, all the spatial path lengths are always balanced.

To perform the discrimination or filtering for a specific set of three nonorthogonal states, with equal *a priori* probabilities, the optimal success probability and output states were calculated following the method in [16]. Using the input and output states in the larger Hilbert space, the corresponding unitary transformation was then calculated and factorized into a sequence of beam splitter transformations. By rotation of the half-wave plates in each of the VBSs the desired transmission and reflectivity were achieved. The experiments were carried out by preparing one of the three nonorthogonal states at a time and measuring the current at the photodiodes PD1 through PD4. For obtaining the probability of an individual photon reaching each detector, the signals at these detectors were normalized to their sum.

In order to demonstrate state discrimination with this experimental setup we examined the set of three nonorthogonal pure states $|\psi_1\rangle = (\sqrt{2/3}, 0, 1/\sqrt{3})$ and $|\psi_{2,3}\rangle = (0, \pm 1/\sqrt{3}, \sqrt{2/3})$. The optimal output states, in the total Hilbert space, are found to be $|\psi_1\rangle_{\text{out}} = (1/\sqrt{3}, 0, 0, \sqrt{2/3})$, $|\psi_2\rangle_{\text{out}} = (0, \sqrt{2/3}, 0, \sqrt{1/3})$, and $|\psi_3\rangle_{\text{out}} = (0, 0, \sqrt{2/3}, \sqrt{1/3})$. The desired unitary transformation was achieved by using two VBSs with the transmission coefficients $t_{14} = t_{34} = 1/\sqrt{2}$ (and $t = 1$ for the rest of the VBSs), and an additional 50/50 beam splitter to couple output rails 2 and 3.

The experimental results are shown in Fig. 2. Figure 2(a) pertains to the case when all states are discriminated. The average probability of success was measured to be 54.5%. The probability of obtaining an erroneous result was about 3%. These errors were mainly the result of imperfect visibility (due to imperfect alignment and angular uncertainty in the wave plate settings), drift, and uncertainty in phase adjustment, where a 12° phase error on one beam corresponds to a 2% error rate. For this set the optimal POVM is predicted to yield conclusive outcomes 55.6% of the time. By comparison, *any* projection valued measurement (PVM) strategy has a success probability of less than 33.3%. This is because the only way a PVM can *guarantee* that we had $|\psi_i\rangle$ is to project onto the unique state orthogonal to all input vectors $|\psi_{j\neq i}\rangle$. In this case, all three such vectors are nonorthogonal, so no orthonormal set of projectors can include more than one of them; no more than one of the three states can be unambiguously distinguished. Given

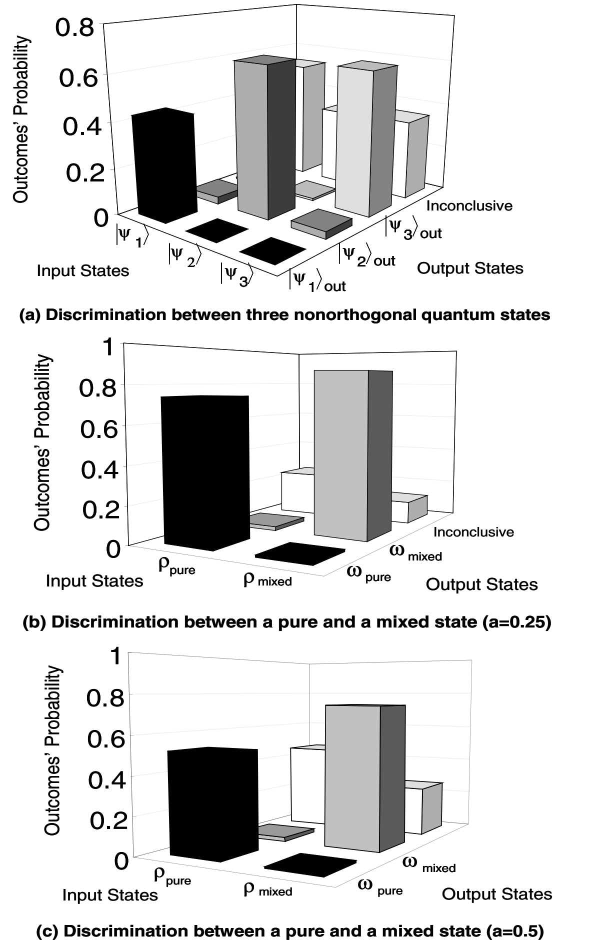

FIG. 2. Experimental data: the results of state discrimination and of state filtering for the cases of $a = 0.25$ and $a = 0.5$, respectively, are presented in parts (a), (b), and (c). Each row corresponds to preparation of a pure or a mixed quantum state. The last column in each figure represents the inconclusive outcomes. The diagonal and off-diagonal elements of other columns represent the successful and erroneous detections, respectively. The probability of each outcome is a measure of the fraction of photons reaching the corresponding detector.

equal *a priori* probabilities, this means that PVMs can succeed no more than 33.3% of the time. In fact, for our input states the optimum PVM is the one that picks out $|\psi_2\rangle$ (or $|\psi_3\rangle$, their success probabilities being equal), with a 25.4% probability of success. In the above example we have shown an improvement of more than a factor of 2 over any possible projective measurement.

For quantum state filtering we considered a family of three-state sets, $|\psi_1\rangle = (\sqrt{1-a^2}, 0, a)$ and $|\psi_{2,3}\rangle = (0, \pm 1/\sqrt{2}, 1/\sqrt{2})$, characterized by the real parameter $a > 0$. The goal was to unambiguously distinguish $|\psi_1\rangle$ from the other two states $\{|\psi_2\rangle, |\psi_3\rangle\}$, each of which has an overlap of $\frac{a}{\sqrt{2}}$ with $|\psi_1\rangle$. The optimal output states are $|\psi_1\rangle_{\text{out}} = (\sqrt{1-a}, 0, 0, \sqrt{a})$ and $|\psi_{2,3}\rangle_{\text{out}} = (0, \pm 1/\sqrt{2}, \sqrt{(1-a)/2}, \sqrt{a/2})$. The unitary transformation for optimal filtering was achieved by beam splitters with pa-

TABLE I. Experimental and theoretical success probabilities of POVMs vs the optimal PVM for State Filtering (SF, for $a = 0.25$ and $a = 0.5$) and State-Discrimination (SD).

	SF ($a = 0.25$)	SF ($a = 0.50$)	SD
$\mathrm{POVM}_{\mathrm{exp}}$	82%	66%	54.5%
$\mathrm{POVM}_{\mathrm{th}}$	83.3%	66.6%	55.6%
$\mathrm{PVM}_{\mathrm{th}}$	64.6%	58.3%	25.4%

rameters $t_{14} = 1/\sqrt{1 + a}$ and $t_{34} = \sqrt{1 - a}$, and $t = 1$ for the other VBSs in the setup. For these sets of states, the optimal success probabilities PVMs and POVMs are $(2 - a^2)/3$ and $1 - (2a)/3$, respectively.

The experiments were performed for $a = 0.25$ and $a = 0.5$. Figure 2(b) pertains to the case $a = 0.25$. The average probability of success for discriminating the state $|\psi_1\rangle$ from $\{|\psi_{2,3}\rangle\}$ was measured to be about 82%, consistent with the theoretical prediction of 83.3%. There was an error rate of about 1.7%. The advantage of the POVM measurements over error-free projective measurements was found to be about 17.4% for this case. For $a = 0.5$, Fig. 2(c), the probability of success was found to be about 66%, with an error probability of less than 1.3%. This can be compared with the theoretical predictions of 66.6%. In this case the advantage over PVMs reduces to 7.7%. As we argued above, these filtering experiments are equivalent to discrimination between the pure state $\rho_1 = |\psi_1\rangle\langle\psi_1|$, with $\eta_1 = 1/3$, and the mixed state $\rho_{23} = (|\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|)/2$, with $\eta_{23} = 2/3$. The experimental results are summarized in Table I.

In conclusion, we have presented the first explicit experimental demonstration that POVMs can achieve a lower inconclusive rate than any projective measurements for unambiguous discrimination between nonorthogonal states, using an optical interferometer to implement arbitrary unitary operations in a four-dimensional Hilbert space. We have demonstrated the first optimal unambiguous discrimination between three linearly independent nonorthogonal pure states, as well as the first experimental realization of unambiguous discrimination between a pure and a mixed quantum state (quantum state filtering). A significant advantage of generalized measurement over projective measurement was observed. Unambiguous state discrimination plays an important role in the field of quantum information processing and has applications to quantum cryptography [2], quantum cloning [23], quantum state separation [24], and entanglement concentration [1,14]. Some quantum information tasks are likely to take advantage of three- or higher-dimensional Hilbert spaces [25], where the advantage of POVMs becomes increasingly significant, as observed here. We believe that generalized optical networks like the one demonstrated here will be of use for a wide variety of small-scale quantum information tasks [22,26] and will prove particularly important for devices such as repeaters and cloners in quantum communications systems.

This work was supported by the DARPA-QuIST program (AFOSR agreement No. F49620-01-1-0468), NSERC, and Photonics Research Ontario. The research of J. B. was supported by ONR. We thank Jeffrey Lundeen, Kevin Resch, Mark Hillery, Ulrike Herzog, and Yuqing Sun for many helpful discussions.

[1] A. Chefles, Contemp. Phys. **41**, 401 (2000).
[2] C. H. Bennett and G. Brassard, in *Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing* (IEEE, New York, 1984), p. 175.
[3] C.W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976).
[4] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987); D. Dieks, Phys. Lett. A **126**, 303 (1988); A. Peres, Phys. Lett. A **128**, 19 (1988).
[5] A. Chefles and S. M. Barnett, J. Mod. Opt. **45**, 1295 (1998).
[6] B. Huttner *et al.*, Phys. Rev. A **54**, 3783 (1996).
[7] S. M. Barnett and E. Riis, J. Mod. Opt. **44**, 1061 (1997).
[8] R. B. M. Clarke *et al.*, Phys. Rev. A **63**, 040305(R) (2001).
[9] R. B. M. Clarke *et al.*, Phys. Rev. A **64**, 012303 (2001).
[10] U. Herzog and J. A. Bergou, Phys. Rev. A **65**, 050305(R) (2002).
[11] T. Rudolph, R.W. Spekkens, and P. S. Turner, Phys. Rev. A **68**, 010301(R) (2003).
[12] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
[13] M. A. Neumark, Izv. Akad. Nauk SSSR, Ser. Mater. **4**, 277 (1940).
[14] A. Chefles, Phys. Lett. A **239**, 339 (1998).
[15] A. Chefles and S. M. Barnett, Phys. Lett. A **250**, 223 (1998); C.W. Zhang, C. F. Li, and G. C. Guo, Phys. Lett. A **261**, 25 (1999).
[16] Y. Sun, J. A. Bergou, and M. Hillery, Phys. Rev. A **66**, 032315 (2002).
[17] J. A. Bergou, U. Herzog, and M. Hillery, Phys. Rev. Lett. **90**, 257901 (2003).
[18] Y. Sun, M. Hillery, and J. A. Bergou, Phys. Rev. A **64**, 022311 (2001) for $N = 3$; J. A. Bergou, M. Hillery, and Y. Sun, J. Mod. Opt. **47**, 487 (2000) for $N = 2$.
[19] M. Reck *et al.*, Phys. Rev. Lett. **73**, 58 (1994).
[20] M. Zukowski *et al.*, Phys. Rev. A **55**, 2564 (1997).
[21] For example, to prepare the state $|\psi\rangle = (1/\sqrt{2}, 1/\sqrt{2}, 0)$, we align the first and second half-wave plates at 22.5° and 0°, so the first PBS acts as a 50/50 beam splitter and the second has transmission 1.
[22] M. Mohseni, J. S. Lundeen, K. J. Resch, and A. M. Steinberg, Phys. Rev. Lett. **91**, 187903 (2003).
[23] L.-M. Duan and G.-C. Guo, Phys. Rev. Lett. **80**, 4999 (1998).
[24] A. Chefles and S. M. Barnett, J. Phys. A **31**, 10097 (1998).
[25] D. Kaszlikowski *et al.*, Phys. Rev. Lett. **85**, 4418 (2000); T. Durt *et al.*, Phys. Rev. A **67**, 012311 (2003).
[26] N. J. Cerf, C. Adami, and P. G. Kwiat, Phys. Rev. A **57**, R1477 (1998).