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Comparison of unitary transforms using Franson interferometry

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Unknown unitary transforms may be compared to each other in a way which makes it possible to obtain an unambiguous answer, indicating that the transforms are different, already after a single application of each transform. Quantum comparison strategies may be useful for example if we want to test the performance of individual gates in a quantum information or quantum computing network. It is then possible to check for errors by comparing the elements to a master copy of the gate, instead of performing a complete tomography of the gate. In this paper we propose a versatile linear optical implementation based on the Franson interferometer with short and long arms. A click in the wrong output port unambiguously determines that the tested gate is faulty. This set-up can also be used for a variety of other tasks, such as confirming that the two transforms do not commute or do not anticommute.

1. Introduction

In order to realize quantum information applications, we need some basic ingredients. First, we need quantum states, used to encode information. Secondly we need to be able to perform operations on the states, shaping them into the desired form. The elementary transforms, for instance, rotations and phase shifts, are usually arranged into more complicated systems—networks. For the reliable performance of the whole network it is of paramount importance that all the elementary gates perform their function as expected. Errors in the individual gates are spread over the whole system leading to an erroneous overall performance. Hence it is of considerable interest to find ways to identify faulty elementary transforms.

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Identifying erroneous transforms can be accomplished in various ways. It is of course possible to map out a transform, using a large number of test particles prepared in properly chosen initial states, i.e. to perform some sort of process tomography [1]. Such a method could, however, be time consuming and would require a lot of resources. It is also not aiming directly at the question we want to answer. The task is to detect an error in the transform, and this can in principle be done by a single measurement. It is enough to obtain an unambiguous measurement result, conveying the information about the dissimilarity of the tested transforms. The requirement of absolute certainty in the answer eliminates minimum error strategies, which are quite popular in state discrimination problems (for a recent review on state discrimination see, e.g. [2]), from our considerations. Previously, we have investigated several methods to compare unitary transforms and also how to compare quantum states, both with unambiguous and with minimum-error methods [3–6]. In all the proposals, a well chosen initial state is used to test the transforms under scrutiny.

In the following we will discuss some further strategies for detecting a difference between two gates. Tests for when two operators commute or anticommute will also be discussed. We will consider a set-up with two interferometers, arranged in series, exploiting both the internal (spin) as well as the spatial degrees of freedom. We compare the proposed strategy with those studied previously and point out some of its advantages and disadvantages.

2. Unambiguous comparison of two unitary transformations

In our treatment we will consider the following situation. We want to know whether two single qubit transformations, which are assumed to be unitary transformations, are different from each other. In the following, we will talk about photons and polarization, but the results are in principle valid for unitary transforms acting on any two-level system. We choose to parametrize the two unknown single qubit transformations as

$$U = \exp(i\varphi_u)\begin{pmatrix} a_u & -b_u \\ b_u^* & a_u^* \end{pmatrix}, \quad V = \exp(i\varphi_v)\begin{pmatrix} a_v & -b_v \\ b_v^* & a_v^* \end{pmatrix}, \tag{1}$$

with $|a_{u(v)}|^2 + |b_{u(v)}|^2 = 1$. Four real parameters are enough to characterize each unitary transform. To test the transforms with respect to their dissimilarity, following [4], we could insert the transforms into an interferometer as shown in figure 1. In this set-up, there are balanced (non-polarizing) beam splitters at the input and the output, and a single photon, with a chosen polarization, is used as a test particle. The transforms U and V act on the polarization degree of freedom of the photon. Depending on whether U and V are different or not, the photon will exit from the interferometer in a certain output port, with a certain polarization [4]. Certain outcomes can be associated with U and V being different *with certainty*. The probability to detect the two transforms as different reads

$$P_{\text{diff}} = \frac{1}{4}\left(\left|\exp(i\varphi_u)a_u - \exp(i\varphi_v)a_v\right|^2 + \left|\exp(i\varphi_u)b_u - \exp(i\varphi_v)b_v\right|^2\right). \tag{2}$$

Averaged over all possible transforms U and V, the probability to detect a difference is 1/2, a result which was obtained in [4]. The one-photon scheme is easy to

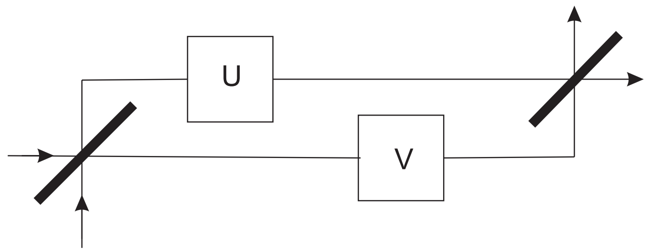

Figure 1. Interferometric comparison involving a single photon test particle, with the transforms connected in parallel.

implement and one is able to detect differences in three out of the four parameters that are used to parametrize the transforms considered. (By repeating the process with a test photon with different polarization we may also detect differences in the remaining parameter.)

The considered one-photon strategy will be optimal in the case that we restrict ourselves to comparing the two transforms by applying each transform only once to a test state with at most one 'particle' (e.g. one photon). It was also shown in [4] that for a single particle test state, entanglement with additional degrees of freedom cannot increase the success probability. On the other hand, allowing the test state to be an entangled two-particle state, in this case, the singlet state, and applying one transform to each particle, increases the average success probability to 3/4. If two applications of the unitary transforms are allowed, however, then two repetitions of the single-photon strategy will do as well as one repetition of the two-particle entangled test state.

The question of general optimality is difficult, since one has so much freedom in choosing how to proceed when comparing two or more given transforms. It seems logical that the transforms have to be applied to a test state and the state then further processed and measured. One has, however, still a lot of possibilities in how to choose the test state and how to apply the transforms to this state. Two choices are 'in parallel' as in the one-photon strategy, or one after the other, 'in series'. In the following section we will discuss another type of comparison set-up, with flexibility as its main advantage, rather that optimal success probability.

3. Comparison using Franson interferometry

The flexibility of the simple interferometric scheme can, however, be developed further. We alter the 'parallel' structure of the interferometer and replace it with an arrangement of two consecutive interferometers 'in series' as shown in figure 2. Also in this case we will use a single photon state with a fixed polarization as the test state. We will refer to this set-up as the Franson interferometer [7] configuration, since it has short and long arms. This interferometer will be used in the time domain. The transforms U and V affect only the polarization degree of freedom. The photon entering the set-up can, in principle, take four possible paths through the interferometer. The first path is the shortest one, without passing through the polarization changing transforms U and V. Photons taking this path arrive at the output beam splitter first. The next two possibilities are the short–long and

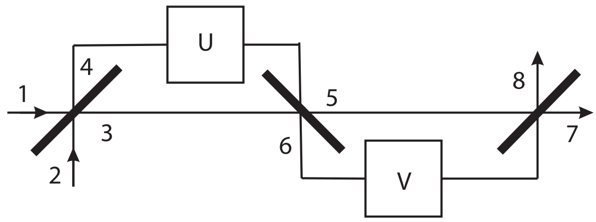

Figure 2. Interferometric comparison involving a single photon and the transforms connected in series.

the long–short paths, where the photon passes through either U or V but not both. When the interferometric set-up is properly aligned, a photon taking either of these two paths will arrive at the output beam splitter at the same time. In the absence of the transforms U and V (or when these are identical), the contributions from these two paths are indistinguishable. The probability amplitudes for the short–long and long–short paths should be added coherently to each other. When the phases in the inteferometer paths are adjusted properly, the two amplitudes can be brought to destructive interference for one of the output ports. The last contribution is coming from the longest path, taking the test photon through both transforms.

With mode indices as for the first beam splitter in figure 2, the non-polarizing beam splitters couple the input and output creation operators according to

$$\hat{a}^{\dagger}_{3,\sigma} = \frac{1}{2^{1/2}}\left(\hat{a}^{\dagger}_{1,\sigma} + \hat{a}^{\dagger}_{2,\sigma}\right), \qquad \hat{a}^{\dagger}_{4,\sigma} = \frac{1}{2^{1/2}}\left(\hat{a}^{\dagger}_{1,\sigma} - \hat{a}^{\dagger}_{2,\sigma}\right). \tag{3}$$

Here $\hat{a}^{\dagger}_{1,\sigma}$ is the creation operator for a photon with polarization σ in mode 1, and similarly for the other modes. Using these relations, and similar ones for the second and third beam splitter, we find that the input creation operators $\hat{a}^{\dagger}_{1(2),\sigma}$ in figure 2, in terms of the output creation operators $\hat{a}^{\dagger}_{7(8),\sigma}$, take the form

$$\begin{aligned}\hat{a}^{\dagger}_{1}(0) = \frac{1}{2(2^{1/2})}\Big[&\exp(i\varphi_{\mathrm{ss}})\hat{a}^{\dagger}_{7,\sigma}(2t) + (\exp(i\varphi_{\mathrm{sl}})U \\ &+ \exp(i\varphi_{\mathrm{ls}})V)\hat{a}^{\dagger}_{7,\sigma}(t+T) - \exp(i\varphi_{\mathrm{ll}})VU\hat{a}^{\dagger}_{7,\sigma}(2T)\Big] \\ + \frac{1}{2(2^{1/2})}\Big[&\exp(i\varphi_{\mathrm{ss}})\hat{a}^{\dagger}_{8,\sigma}(2t) + (\exp(i\varphi_{\mathrm{sl}})U \\ &- \exp(i\varphi_{\mathrm{ls}})V)\hat{a}^{\dagger}_{8,\sigma}(t+T) + \exp(i\varphi_{\mathrm{ll}})VU\hat{a}^{\dagger}_{8,\sigma}(2T)\Big]\end{aligned} \tag{4}$$

and

$$\begin{aligned}\hat{a}^{\dagger}_{2,\sigma}(0) = \frac{1}{2(2^{1/2})}\Big[&\exp(i\varphi_{\mathrm{ss}})\hat{a}^{\dagger}_{7,\sigma}(2t) - (\exp(i\varphi_{\mathrm{sl}})U \\ &- \exp(i\varphi_{\mathrm{ls}})V)\hat{a}^{\dagger}_{7,\sigma}(t+T) + \exp(i\varphi_{\mathrm{ll}})VU\hat{a}^{\dagger}_{7,\sigma}(2T)\Big] \\ + \frac{1}{2(2^{1/2})}\Big[&\exp(i\varphi_{\mathrm{ss}})\hat{a}^{\dagger}_{8,\sigma}(2t) - (\exp(i\varphi_{\mathrm{sl}})U \\ &+ \exp(i\varphi_{\mathrm{ls}})V)\hat{a}^{\dagger}_{8,\sigma}(t+T) - \exp(i\varphi_{\mathrm{ll}})VU\hat{a}^{\dagger}_{8,\sigma}(2T)\Big].\end{aligned} \tag{5}$$

Here $\varphi_{ss}, \varphi_{sl}$ etc. are the adjustable phase shifts for the short–short, short–long etc. paths, and U and V affect the polarization state of the photon. The time $2t$ is the time it takes for the photon to pass through the short–short path, and $t + T$ and $2T$ are the times for the short–long (and long–short) and long–long paths, respectively.

In the following, we will assume that we feed the interferometer only from one of the input ports, say mode 1. What can we learn from the output of the interferometer? The output signal in modes 7 and 8 carries information about the tested transforms, distributed over three pulses. The first seems to be of no great interest, as it obviously carries no information about U and V. It could of course be 'recycled' and coupled back into the interferometer. We shall see below, however, that it is possible to use this first pulse also in a more straightforward way. The second and third pulses are important, as they passed through the transforms U and V, so that their polarization was affected. These two pulses are indistinguishable in time, and by adjusting the interferometer, we can make the two spatial components precisely cancel each other in one of the output ports, say mode 8, provided U and V are identical. The photon will always exit in mode 7. When U and V are not affecting the polarization in an identical way, however, the two components will not cancel each other completely in output mode 8. The resulting non-zero probability to detect the photon in this output port, at this time window, is an unambiguous indicator of the two transforms being different. This is very similar to the spirit of the interferometer set-up in [4]. As in the previous work, we should note that detecting the photon in output mode 7 obviously does not allow us to conclude that the transforms were the same. On average, we will have a probability of 1/4 to detect a difference between U and V. In half of the cases, the photon will take the short–short or the long–long path. The figure 1/4 is obtained when averaging over all possible transforms U and V as is also done in [4].

There is, however, an additional interesting point. The third pulse arriving at a time $2T$ later than the first carries information about the product of the transforms, VU. This information can be used further. We can use the first and the last pulse to compare VU (the last pulse) to the identity operation (the first pulse). We compensate for the time difference and bring the pulses into interference for the comparison test. This could be done using a fourth beam splitter. When the measurement result indicates dissimilarity, then the combination of the two transforms could not have been the identity operation. Several transforms which are of interest for quantum information are cyclic. Applying a cyclic transform T a certain number of times results in the identity operation, i.e. $T^k = 1$, where k is the order of the transform. Two examples of cyclic transformations are the Hadamard transform for single qubits and the C-NOT gate for two-qubit transforms. If $U = V$, what we get is a test of whether U^2 equals the identity operator or not.

We could equally well test whether the combination UV differs from the identity operation without making use of the first pulse, only looking at the long–long pulse on its own. In this case, we have to check whether the polarization of the test photon has changed. (Since there is nothing the long–long pulse can interfere with, where the photon exits will not carry any information.) If we are only interested in this information, and not at all in comparing U and V, it would, however, be easiest, and give a higher success probability, not to use an interferometer at all, but to apply U and V in series to our test photon straightaway, and then to measure whether its polarization was changed or not. If we obtain a polarization state orthogonal to the input polarization state, the transforms cannot have been cyclic.

4. Tests of commutativity and anticommutativity

The time ordered sequences of splitter pulses can be used also for additional tests. If we do not balance the interferometer arms in such a way that the long–short and the short–long paths have equal length, then we obtain four time separated pulses at the output. These pulses can then be used for further tests, involving higher order combinations of the operators. For instance, we could ask whether it is possible to check if our two transforms commute. A natural way to check whether U and V commute would be to apply UV in one arm and VU in the other, using an interferometer similar to the one in figure 1. This requires two identical copies each of U and V. If we only have access to one copy each, we have to 'recycle' the test photon. The test strategies for commutativity we will consider below have lower success probability than the 'natural' strategy, their advantage being that they are experimentally simpler in the case that we only have access to one copy each of the transforms.

A simple recycling loop using the output from the second unused output port is shown in figure 3. After the second passage we obtain sixteen components, since each of the four first-order pulses now splits into four. Some of these will interfere. It is also possible to do a time gating after the first passage and recycle only those pulses that we wish to test in the second order. For example, we may feed back only the long–short and the short–long first-order pulses into the interferometer. Each pulse will again be split into four components in both arms. At the output we thus obtain eight terms after the second passage. Two of these will interfere, resulting in seven pulses arriving at seven different times. The seven pulses in one of the outputs will be proportional to

$$\begin{aligned}&\exp[i(\varphi_{ss}+\varphi_{sl})]U, \quad \exp[i(\varphi_{ss}+\varphi_{ls})]V,\\ &\exp[i(\varphi_{sl}+\varphi_{ls})](UV-VU), \quad \exp[2i\varphi_{ls}]V^2, \quad \exp[2i\varphi_{sl}]U^2,\\ &\exp[i(\varphi_{ll}+\varphi_{ls})]VUV, \quad \exp[i(\varphi_{ll}+\varphi_{sl})]VUU.\end{aligned} \tag{6}$$

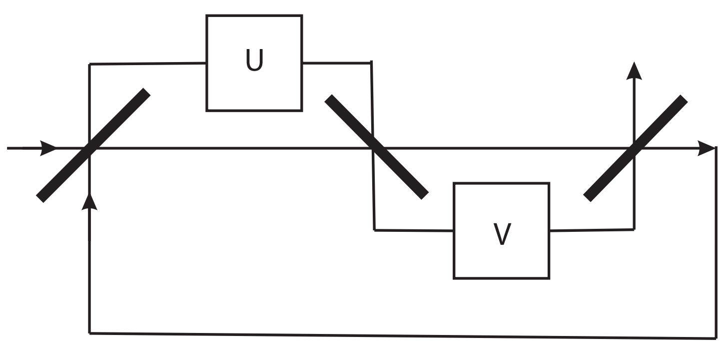

Figure 3. Interferometric comparison involving a single photon and two interferometers connected in series, with recycling of the photon.

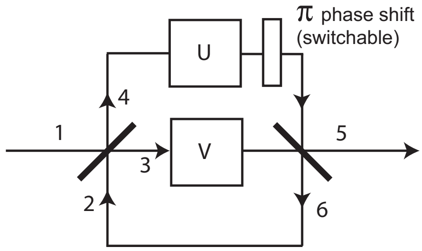

Figure 4. Testing whether two transforms commute or anticommute with an interferometric set-up. The photon is recycled so that one of the output ports (mode 6) is connected into one of the input ports (mode 2). The phase shift of π can be turned on during the second passage in order to reverse the output modes, so that we can test either for commutativity or anticommutativity in the 'free' output port.

The $UV - VU$ term vanishes when the two terms UV and VU are interchangeable, i.e. when the transforms commute. The presence of the photon in the interference term between these pulses would then unambiguously indicate that the transforms did not commute. The other output will contain a term where $UV + VU$ appears, and this can in a similar way be used to test whether the transforms do not anticommute. If a single photon is used as a 'test particle', it can obviously be detected only once, and thus we can either obtain the knowledge that 'U and V definitely do not commute' or 'U and V definitely do not anticommute'. It should also be stressed that with this set-up, it is not possible to unambiguously confirm that the transforms commute, or that they anticommute. This is analoguous to the results on state and transformation comparison obtained in [4, 5]. We can conclude that a set of states or transforms were different, with no error in our statement, just with one run of the comparison strategy. Unless we have more information about the states or the transforms, however, we can never conclude, without error, that they are identical.

An experimentally simpler way to test whether U and V commute or anticommute would be to use an interferometer similar to that in figure 1, but connecting one of the output ports to the unused input, as in figure 4. Here the paths going through U and V should be of different length, so that there is no interference at the second beam splitter after the first passage through the set-up. After the second passage, only the terms which correspond to passing through U in the first round and V in the next, or vice versa, will arrive at the second beam splitter at the same time and interfere. One output mode will now contain a term where $UV - VU$ appears, and the other a term with $UV + VU$. The switchable phase shift of π is used in the second passage, to swap the outputs so that we can test for either commutativity or anticommutativity in the 'free' output port, without need for other switching. Without the π phase shift, after two passages, the creation operator for a photon in input mode 1 will transform according to

$$\begin{aligned}\hat{a}^\dagger_{1,\sigma}(0) \to \frac{1}{2}\bigg\{ & U\hat{a}^\dagger_{5,\sigma}(t_u) + V\hat{a}^\dagger_{5,\sigma}(t_v) \\ & + \frac{1}{2}\big[U^2\hat{a}^\dagger_{5,\sigma}(2t_u) + V^2\hat{a}^\dagger_{5,\sigma}(2t_v) - (UV + VU)\hat{a}^\dagger_{5,\sigma}(t_u + t_v) \\ & - U^2\hat{a}^\dagger_{6,\sigma}(2t_u) + V^2\hat{a}^\dagger_{6,\sigma}(2t_v) + (UV - VU)\hat{a}^\dagger_{6,\sigma}(t_u + t_v)\big]\bigg\},\end{aligned} \tag{7}$$

where $t_u(t_v)$ is the time it takes to pass one time through the $U(V)$ path. The terms on the first line will exit the interferometer after the first passage and the rest of the terms after the second passage. At the time $t_u + t_v$, it is possible to test for commutativity in output mode 6 and for anticommutativity in output mode 5. If we want to test for commutativity in output mode 5, we can apply a phase shift of π during the second passage in order to swap the output modes.

This set-up contains one interferometer loop less than the one in figure 3 and would be somewhat easier to align. Also, many of the unwanted photon paths are eliminated, so that the success probability of detecting non-commutativity is four times as high as for the Franson set-up in figure 3. With more switching, it would be possible to further enhance the success probability, for example by recycling *both* outputs, using also the input port where the photon first entered the interferometer. From an experimental point of view, however, a fixed set-up is easier to implement.

In a similar way we can test higher powers of the individual operators U and V, up to fourth order after the second passage. One only needs to balance the arm lengths and phases in order to make the corresponding splitter pulses overlap as desired. Both set-ups, as shown in figures 3 and 4, can be used for tests of higher powers of operators U and V. A test of higher order powers of a transform is useful, when we have reason to believe that the transforms differ only slightly. The repeated application of the transforms makes the possible difference larger and increases the probability to detect the difference. This is related to the Grover search algorithm [8]. Somewhat simplified, we can argue as follows. A given input state, say $|0\rangle$, is mapped, by application of the oracle, onto the state $\cos\theta|0\rangle + \sin\theta|i\rangle$, where $|i\rangle$ is one of a set of states $\{|i\rangle\}$, $i = 1, 2, \ldots, n$, and the goal is to determine which one it is. Repeated application of the oracle k times gives the state $\cos k\theta|0\rangle + \sin k\theta|i\rangle$. To maximize the probability of obtaining the answer i, one should keep on iterating until $k\theta \approx \pi/2$. Provided the optimal number of oracle applications can be well estimated, the probability to obtain the desired signal (by a measurement in the basis $\{|0\rangle, |1\rangle, |2\rangle, \ldots\}$) will be large.

In contrast, if we try to determine i after each step, the probability to succeed will be lower. After each step, the probability to obtain the desired signal is $p_i = \sin^2\theta$. After k such independent tests, the probability to succeed is approximately

$$p_i^{(k)} = 1 - (\cos^2\theta)^k \approx k\sin^2\theta \approx k\theta^2, \tag{8}$$

if we assume that θ is small. This figure is to be compared with

$$p_i^{(k)} = \sin^2 k\theta \approx k^2\theta^2, \tag{9}$$

which is the success probability if we apply the oracle, or the transforms, repeatedly and measure only after k applications, not after each application. Here we have assumed that also $k\theta$ is small. We should, however, keep in mind that we must be able to make a good estimate of the ‘smallness’ of the difference between the transforms. When the transforms are completely unknown, they may be very different, and in this case it is better to measure after each application of the transforms. One way to apply the same transforms many times, is to build a ring resonator, where the signals would experience the transformations many times and then be coupled out of the resonator for measurement.

5. Conclusions

In this paper we have discussed comparison of unitary transforms. With the Franson set-up, it is possible both to compare two transforms to each other, and to test if application of both transforms in series is different from the identity operator, that is, if $UV \neq 1$. Let us note that we could achieve similar results by tossing a coin, and, if we obtain tails, we realize a comparison strategy using an interferometer with U and V in one arm each, as in figure 1. If we obtain heads, we use an interferometer with U and V in the same arm, thus comparing UV to the identity operator. We have also discussed how to determine that two unitary transforms do not commute, or that they do not anticommute. The main advantage of the Franson set-up is its flexibility; it can be used for all the tasks discussed in this paper. It will not always result in optimal success probability. We have given some examples of other set-ups, such as the ones in figures 1 and 4, which will yield a higher success probability for a special selected task. General statements regarding absolute optimality are difficult to make, since there is so much freedom in choosing how to compare two given unitary transforms. But the set-up in figure 1 is, for example, optimal for comparison of U and V in the case that each transform can be applied only once to a test state of only one photon [4].

The experimental realizations suggested in this paper are for the case when U and V are single-qubit transforms. It is possible to use similar strategies to test e.g. two-qubit transforms, using two test particles. The test state could here be either separable or entangled. In our previous work, we have discussed strategies with entangled test states [4]. It turns out that using an entangled state often can enhance the success probability of detecting a difference in the transforms. It is interesting to note that a set-up using feedback in a way somewhat similar to our set-ups, has been used for photon-number-resolving detection [9]. Finally, let us point out that the tasks discussed could be realized also using coherent beams instead of single photons. In this case, problems related to detector efficiencies could easily be circumvented.

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