

which means

$$\epsilon_{\text{corr}} = 2\pi^{-2}(1 - \ln 2)\ln r_s + \text{const} \quad r_s \rightarrow 0 \quad (12.61)$$

This result is originally due to Macke.<sup>1</sup> Note that this expression is nonanalytic in  $r_s$  and has no power series around  $r_s = 0$ .

The constant term in the correlation energy requires the evaluation of all the remaining terms in Eq. (12.26). In particular, it is essential to prove that the arbitrary wavenumber  $q_c$  drops out of the final answer for  $\epsilon_c$ . This calculation is very similar to that in Sec. 30 and will not be repeated here. Furthermore, it is easy to see that  $E_2^c$  and  $E_2^d$  vanish identically, while

$$E_2^b = \frac{Ne^2}{2a_0} \epsilon_2^b$$

is just the second-order exchange energy studied in Prob. 1.4. The final expression can only be obtained numerically, and the correlation energy becomes

$$\begin{aligned} \frac{E_{\text{corr}}}{N} &= \frac{e^2}{2a_0} \left[ \frac{2}{\pi^2} (1 - \ln 2)\ln r_s - 0.094 + O(r_s \ln r_s) \right] \\ &= \frac{e^2}{2a_0} [0.0622 \ln r_s - 0.094 + O(r_s \ln r_s)] \end{aligned} \quad (12.62)$$

correct through order  $\ln r_s$  and  $r_s^0$ .<sup>†</sup> By an extension of the arguments presented here, DuBois<sup>2</sup> shows that the sum of the next most divergent terms in each order in perturbation theory [those terms with one less power of  $U_0(q)$ ] gives a correction of  $O(r_s \ln r_s)$  to Eq. (12.62).

#### EFFECTIVE INTERACTION

We have already mentioned that the perturbation expansion fails because of the singular  $(\mathbf{q})^{-2}$  behavior of  $U_0(\mathbf{q})$ . In contrast, the ring approximation to the effective interaction  $U_r(\mathbf{q}, q_0)$  has a very different behavior at long wavelengths. For simplicity, we shall consider only the static limit ( $q_0 = 0$ ), and a combination of Eqs. (12.24) and (12.49) yields

$$U_r(\mathbf{q}, 0) = \frac{4\pi e^2}{q^2 + (4\alpha r_s/\pi) k_F^2 g(q/k_F)} \quad (12.63)$$

<sup>1</sup> W. Macke, *Z. Naturforsch.*, **5a**:192 (1950).

<sup>†</sup> The logarithmic term was first obtained by W. Macke, *loc. cit.*, and the complete expression was then derived by M. Gell-Mann and K. A. Brueckner, *loc. cit.* See also L. Onsager, L. Mittag, and M. J. Stephen, *Ann. Physik*, **18**:71 (1966).

<sup>2</sup> D. F. DuBois, *Ann. Phys. (N.Y.)*, **7**:174, appendix C, (1959). DuBois' calculation was repeated and corrected by W. J. Carr, Jr., and A. A. Maradudin, *Phys. Rev.*, **133**:A371 (1964), who find  $0.018r_s \ln r_s$  as the next correction to  $\epsilon_{\text{corr}}$ .

where

$$g(x) = \frac{1}{2} - \frac{1}{2x} (1 - \frac{1}{4}x^2) \ln \left| \frac{1 - \frac{1}{2}x}{1 + \frac{1}{2}x} \right| \quad (12.64)$$

Thus the medium composed of the electrons and the positive background modifies Coulomb's law. It is clear from Eq. (12.63) that this modification is important only for wavelengths  $(q/k_F)^2 \leq r_s$ ; in the high-density limit where  $r_s \rightarrow 0$ , we can therefore approximate  $g(q/k_F)$  by  $g(0) = 1$ , so that

$$U_r(\mathbf{q}, 0) \approx \frac{4\pi e^2}{r_s \rightarrow 0 \quad q^2 + (4\alpha r_s / \pi) k_F^2} \quad (12.65)$$

Hence the effective potential is cut off for  $q^2 \lesssim r_s k_F^2$  and is *finite* at  $q = 0$ , which confirms the assertions below Eq. (12.23). This behavior provides a physical basis for the cutoff used to find  $\epsilon_{\text{corr}}$  in Eq. (12.60) and in Prob. 1.5.

Although Eq. (12.65) is only an approximation to the exact  $U_r(\mathbf{q}, 0)$  given in Eq. (12.63), it is very easy to take the Fourier transform of this approximate expression, which gives a Yukawa potential. We thereby obtain a qualitative picture of the effective interaction in coordinate space

$$V_r(\mathbf{x}) \approx e^2 e^{-q_{TF} x} x^{-1} \quad (12.66)$$

Hence the simple  $e^2/x$  Coulomb's law between two charges is "shielded" with the Thomas-Fermi<sup>1</sup> screening length  $q_{TF}^{-1}$  defined by

$$q_{TF}^2 = \frac{4\alpha r_s}{\pi} k_F^2 = \frac{4}{\pi} \left( \frac{4}{9\pi} \right)^{\frac{1}{2}} r_s k_F^2 = 0.66 r_s k_F^2 \quad (12.67)$$

In fact, the nonanalytic structure of (12.63) complicates the actual expression for  $V_r(\mathbf{x})$  considerably, as is discussed in detail in Sec. 14.

In the present section,  $U_r(\mathbf{q}, q_0)$  has been used only to evaluate the correlation energy, which is an equilibrium property. As shown in the preceding paragraph, however,  $U_r$  contains much additional physical information because it determines the effective static and dynamic interparticle potential. This behavior is really a particular example of the response to an external perturbation. For this reason, we shall first develop the general theory of linear response (Chap. 5) and then return to the nonequilibrium properties of the degenerate electron gas.

## PROBLEMS

**4.1.** A uniform spin- $s$  Fermi system has a spin-independent interaction potential  $V(\mathbf{x}) = V_0 x^{-1} e^{-x/a}$ .

(a) Evaluate the proper self-energy in the Hartree-Fock approximation. Hence find the excitation spectrum  $\epsilon_{\mathbf{k}}$  and the Fermi energy  $\epsilon_F = \mu$ .

<sup>1</sup> The Thomas-Fermi theory is described in Sec. 14.