Integrals and Antiderivatives

1. **Definition of Definite Integral**

If f is a function defined on [a, b], the definite integral of f from a to b is the number (x_i^* are sample points in the subintervals $[x_{i-1}, x_i]$. These sample points could be left endpoints or right endpoints or any numbers between the endpoints)

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided that this limit exists. If it does exist, we say that f is integrable on [a, b].

2. **Theorem** (it is much simpler to use than Definition of Infinite Integral

   If f is integrable on [a, b], then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Example: Evaluate $\int_0^3 (x^3 - 6x)$

Solution: $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$

Thus, $x_0 = 0, x_1 = \frac{3}{n}, x_2 = \frac{6}{n} x_3 = \frac{9}{n},$ and, in general, $x_i = \frac{3i}{n}$ Since we are using right endpoints, we can use Theorem 4:

$$\int_0^3 x^3 - 6xdx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\frac{3}{n}$$

$$=\lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left[(\frac{3i}{n})^3 - 6(\frac{3i}{n})\right]$$

$$=\lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n} \left[\frac{27}{n^3} i^3 - \frac{18i}{n})\right]$$

$$=\lim_{n\to\infty} \frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54\sum_{i=1}^{n} i}{n^2}$$

$$=\lim_{n\to\infty} \left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n^2}\frac{n(n+1)}{2}\right\}$$

$$=\lim_{n\to\infty} \left[\frac{81}{4}(1+\frac{1}{n})^2 - 27(1+\frac{1}{n}\right]$$

$$=\frac{81}{4} - 27 = \frac{-27}{4}$$

Practice:

1. $\int_1^2 y^2 + y^{-2}dy$
2. $\int_4^0 \sqrt{t}(t-2)dt$

Solutions:

1. $\frac{17}{6}$
2. $-\frac{32}{15}$

Indefinite Integral

1. the notation $\int f(x)\,dx = F(x)$ is traditionally used for an antiderivative of f and is called an indefinite integral.

   Thus $\int f(x)\,dx = F(x)$ Means $F' = f(x)$

Note: You should distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(\mathbf{x})\,\boldsymbol{dx}$ is a number, whereas an indefinite integral $\int f(\mathbf{x})\,\boldsymbol{dx}$ is a function (or family of functions).

Example

$\int \cos dx = sinx$ $\qquad$ $\int \frac{1}{x} = \ln(x)$

Example

Find the general indefinite integral $\int (10x^4 - 2\,sec^2x)dx$

Solution: $\int (10x^4 - 2\,sec^2x)dx = 10\int x^4dx - 2\int sec^2x$

$= 10\frac{x^5}{5} - 2tanx + c$

$= 2x^5 - 2tanx + c$

2. Table of Indefinite Integral

$\frac{d}{dx}x^n = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\cot x = -\csc^2 x$	$\int \csc^2 x\,dx = -\cot x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\int \csc x \cot x\,dx = -\csc x + C$

Antiderivative

Definition: A function F is called an antiderivative of f on an interval of I if $F' = f(x)$ for all x in I.

Theorem: if F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x)+C$$

Where C is an arbitrary constant.

Table of antiderivative formulas:
See Above (Same as Table of Indefinite Integral Table)

Example: Find f if $f' = x\sqrt{x}$ *and* $f(1) = 2$

Solution: The general antiderivative of $f'(x) = x^{\frac{3}{2}}$ is $f(x)=\frac{x^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5}x^{\frac{5}{2}}+C$

Solving for C, we can use the fact that $f(1) = 2$

$f(1) = 2/5+C=2$
Solving for C, we can get C=8/5

So, the solution is $f(x)=\frac{2x^{\frac{5}{2}}+8}{5}$

Practice:
1. Find the antiderivative of the following function:
    a. $f(x) = 3\cos(x) + \frac{2x^4-\sqrt{x}}{x}$
    b. $f(x) = (x+1)(2x-1)$

Solutions:
1.
    a. $3\sin(x) + \frac{1}{2}x^4 - 2\sqrt{x} + C$
    b. $\frac{2x^3}{3} + \frac{x^2}{2} - x + C$