Intermediate Value Theorem

Definition: Suppose that f is continuous on the closed interval [a, b] and let N be any number between f(a) and f(b), where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that f(c)=N

Example: Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.
Solution: Let f(x)= $4x^3 - 6x^2 + 3x - 2 = 0$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that f(c)=0. Therefore, we take a =1, b=2, and N=0. we have

$$f(1) = 4-6+3-2=-1 < 0$$
$$f(2) = 32-23+6-2=12 > 0$$

Thus f (1) <0< f (2) that is, N=0 is a number between and f (1) and f (2). Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says there is a number c between 1 and 2 such that f(c)=0. In other words, the equation has at least one root in the interval (1, 2).

Example: Is there a solution to $x^5 - 2x^3 - 2 = 0$ between $x = 0$ & $x = 2$?
Solution: At x=0

$$f(x) = (0)^5 - 2(0)^3 - 2 = -2$$

At x=2

$$f(x) = (2)^2 - 2(2)^3 - 2 = 14$$

Now, we know that at x=0, the curve is below zero and at x=2 the curve is below zero.
And, working with a polynomial, the curve will be continuous, so somewhere in between, the curve **must** cross y=0. In conclusion, yes, there is a solution to $f(x) = x^5 - 2x^3 - 2 = 0$ on the interval [0,2].

Practice:

1. How many zeros does $f(x) = 55x^3 - 60x^2 + 20x - 2$ have between $x = 0$ and $x = 6$?
2. How many zeros does $f(x) = x^4 - 3x^3 + 2x^2 - 0.1$ have between -0.5 and 2.5?
3. When given $f(x) = \frac{3x+1}{x^2-3}$, how many solutions are there to the equation $f(x) = 0$?

Solutions:

1. 3
2. 4
3. 1