Limit

1. Definition of limit

Suppose f(x) is defined when x is near the number a. (This means that f is defined on some open interval that contains a, expect possibly at a itself.)

Then we write $\lim_{n\to\infty} f(x) = L$

And say "the limit of f(x), as x approach a, equals L"

If we can make the values of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

2. Precise Definition of Limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. Then we say the limit of f(x) as x approaches a is L, and we write

$$\lim_{x\to\infty} f(x) = L$$

If for every number ε > 0, there is a corresponding number δ >0 such that

If o<|x-a|< δ then. | f(x)-L| < ε

3. Calculate the limit
   I. Direct substitution property: If f is a polynomial or a. rational function and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$

Example1: Prove $\lim_{n\to\infty}(4x-5) = 7$

Solution: Let ε be a given positive number. According to Precise Definition with a=3, and L=7, we need to find a number δ such that

If o<|x-3|< δ then | (4x-5)-7| < ε
But | (4x-5)-7|=| 4x-12|=4|x-3|. Therefore, we want
If o<|x-3|< δ then 4|x-3| < ε
Now we notice that 4|x-3| < ε, then |x-3| < ε/4, so let's choose δ= ε/4
So, we can write the following:
o<|x-3|< δ then 4|x-3| < 4δ= ε

Example2 Evaluate $\lim_{h\to 0}\frac{(3+h)^2-9}{h}$

Solution: We can simply the function since we cannot use direct substitution.

F(h)= $\lim_{h\to 0}\frac{(3+h)^2-9}{h}=\frac{(9+6h+h^2)-9}{h}=\frac{6h+h^2}{h}=6+h$

Example 3 $\lim_{x\to 2}x^2+2x$

Solution: Direct Substitution $\lim_{x\to 2}(x^2+2x)=2x2+2x2=8$

Limits involving infinity

Definition 1:

The notation $\lim_{x\to \mathrm{a}}f(x)=\infty$ means that the values of f(x) can be made arbitrarily large (as large as we please) by taking x sufficiently close to a (either side of a) but not equal to a.

Definition 2:

The line x=a is called a vertical asymptote of the curve y= f(x) if at least one of the following statements is true:

$\lim_{x\to \mathrm{a}}f(x)=\infty$ $\lim_{x\to a^-}f(x)=\infty$ $\lim_{x\to a^+}f(x)=\infty$

$\lim_{x\to \mathrm{a}}f(x)=-\infty$ $\lim_{x\to a^-}f(x)=-\infty$ $\lim_{x\to a^+}f(x)=-\infty$

Definition 3:

The line y=L is called a horizontal asymptote of the curve y= f(x) if either

$\lim_{x\to\infty}f(x)=L$ $\lim_{x\to-\infty}f(x)=L$

Definition 4:

The notation $\lim_{x\to\infty}f(x)=\infty$is used to indicate that the value of f(x) becomes larger as x becomes large. Similar meanings are attached to the following symbols:

$\lim_{x\to-\infty}f(x)=-\infty$ $\lim_{x\to\infty}f(x)=-\infty$ $\lim_{x\to-\infty}f(x)=\infty$

Example 1

Find $\lim_{x\to 3^+}\frac{2x}{x-3}$ and $\lim_{x\to 3^-}\frac{2x}{x-3}$

Solution:

If x is close to 3 but larger than 3, then the denominator x-3 is a small positive number and 2x is close to 6. So, the quotient $\frac{2x}{x-3}$ is a large positive number. Thus, intuitively, we see that

$$\lim_{x\to 3^+}\frac{2x}{x-3}=\infty$$

Likewise, if x is close to 3 but smaller than 3, then is a small negative number but 2x is still a positive number (close to 6). So $\frac{2x}{x-3}$ is a numerically large negative number. Thus

$$\lim_{x\to 3^+}\frac{2x}{x-3}=-\infty$$

Therefore x=3 is vertical asymptote.

Example 2 Find $\lim_{x\to\infty}\frac{x^2-1}{x^2+1}$

Solution:

$$\lim_{x\to\infty}\frac{x^2-1}{x^2+1}=\lim_{x\to\infty}\frac{x^2+1-2}{x^2+1}=\lim_{x\to\infty}1-\frac{2}{x^2+1}$$

Since x approach infinity, so $\lim_{x\to\infty}1-\frac{2}{x^2+1}=1$

Therefore, y= 1 is horizontal asymptote.

Example 3 $\lim_{x\to\infty}(x^2-x)$

Note: It would be wrong to write

$$\lim_{x\to\infty}(x^2-x)=\lim_{x\to\infty}x^2-\lim_{x\to\infty}x=\infty-\infty=0$$

Solution:

The Limit Laws can’t be applied to infinite limits because∞ is not a number ($\infty-\infty$)can’t be defined). However, we can write

$$\lim_{x\to\infty}(x^2-x)=\lim_{x\to\infty}x(x-1)=\infty$$

Continuity

Definition 1:

A function f(x) is continuous at a number a if $\lim_{x\to a} f(x) = f(a)$

Notice that Definition implicitly requires three things if f is continuous at a:

1. $f(a)$is defined (that is, a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Definition2:

A function f is continuous from the right at a if

$$\lim_{x\to a^+} f(x) = f(a)$$

And f is continuous from the left at a if

$$\lim_{x\to a^-} f(x) = f(a)$$

Definition3:

A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Theorem: The following types of functions are continuous at every number in their domains: polynomials, rational functions, root functions, trigonometric functions.

Example: Show that the function f(x)=1-$\sqrt{1-x^2}$ is continuous on the interval [-1,1]

Solution: -1<a<1, then using the Limit Laws, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a}(1\text{-}\sqrt{1-x^2}) = 1 - \lim_{x\to a}(1\text{-}\sqrt{1-x^2})$$

$$= \sqrt{\lim_{x\to a}(1-x^2)} = 1 - \sqrt{1-a^2} = f(a)$$

Thus, by Definition l, f is continuous at a if -1<a<1. Similar calculations show that

$$\lim_{x\to -1^+} f(x) = 1 = f(-1) \qquad \lim_{x\to -1^-} f(x) = 1 = f(1)$$

and so f is **continuous** from the right at -1 and continuous from the left at 1. Therefore, according to Definition 3, is continuous on [-1,1].

(The graph of is function is the lower half of the circle)

Differentiability

1. Differentiation is the process of finding a derivative

2. Connection between differentiation and continuity
   Any differentiable function must be continuous at every point in its domain.
   The converse does not hold: a continuous
   function need not be differentiable. For example, a function with a bend, cusp, or vertical tangent may be continuous, but fails to be differentiable at the location of the anomaly.

3. By the definition of derivative, it is easy to see

$$f'(\mathrm{x}) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$f'(\mathrm{x}) = \lim_{x \to a} \frac{f(x) - f(a)}{a}$$

4. Property of derivative

Multiplication by a constant Note: Derivative of a constant, c, equals 0.	$f(x) = cf$	$f'(x) = cf'$
Power Rule	$f(x) = x^n$	$f(x) = (n)x^{n-1}$
Sum Rule	$f + g$	$f' + g'$
Difference Rule	$f - g$	$f' - g'$
Product Rule	fg	$fg' + gf'$
Quotient Rule	$\frac{f}{g}$	$\frac{gf' - fg'}{g^2}$
Reciprocal Rule	$\frac{1}{f}$	$\frac{(-f')}{f^2}$

Chain Rule (Composition)	$f \circ g$	$(f \circ g) \times g$
Chain Rule using (")	$f(g(x))$	$f'(g(x))g'(x)$
Chain Rule using $\frac{d}{dx}$	$\frac{dy}{dx}$	$\frac{dy}{du}\frac{du}{dx}$

Examples: differentiate

1. f(x)=cos(x) $f' = -sinx$
2. $x^8 + x^5 - 6x - 5 = 8x^7 + 5x^4 - 6$
3. $\frac{d}{dx}(3x^4) = 3\frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$
4. Find the derivative of f(x)=$(x^2 + 2x)$ at x=2

$$f'(\mathrm{x}) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{(a+h)^2 + 2(a+h) - (a^2 + 2a)}{h}$$
$$= \frac{a^2+2ah+h^2+2a+2h-a^2-2a}{h} = \frac{2ah+h^2+2h}{h} = 2a + h + 2 = 6$$

Practice Problems:

Limits:

1. Find the limit of the following:
   a. $\lim_{x \to 10} \frac{x}{2}$
   b. $\lim_{x \to 1} \frac{x^2-1}{x-1}$
   c. $\lim_{x \to \infty} \frac{5x^2+1}{3x^2-x}$
   d. $\lim_{x \to \infty} 2x - 7x^3$

Definition of derivative practice

1. Use the definition of the derivative to find f ' (-1), where $f(x) = 2x^2 + 1$.

2. Use the definition of the derivative to find f ' (x), where $f(x) = \frac{1}{\sqrt{(x-4)}}$.

Review Derivative Rules

1. Differentiate the following functions

   a. $f(x) = 4x^3$

   b. $f(x) = 4sin(x)cos(x)$

c. $f(x) = (3x+1)^2(x^2+2)$

d. $f(x) = \frac{x^3}{(2x-1)^2}$

Solutions:

Limits:

1.

a. 5

b. 2

c. $\frac{5}{3}$

d. $-\infty$

Definition of Derivative practice:

1. $f'(-1) = -4$
2. $f'(x) = -\frac{1}{2(x-4)^{\frac{3}{2}}}$

Review of Derivative Rules:

1.

a. $f'(x) = 12x^2$

b. $h'(x) = 4\cos^2 x - 4\sin^2 x$

c. $h'(x) = 36x^3 + 18x^2 + 38x + 12$

d. $g'(x) = \frac{2x^5(4x-3)}{(2x-1)^3}$