The Mean Value Theorem
(Fundamental Theorem of Calculus)

Definition:

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval [a, b].
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or equivalently, $(b) - f(a) = f'(c)(b - a)$

Example: To illustrate the Mean Value Theorem with a specific function, let's consider $f(x) = x^3 - x, a = 0, b = 2.$

Solution: Since f is a polynomial, it is continuous and differentiable for all x, so it is certainly continuous on [0, 2] and differentiable on (0, 2). Therefore, by the Mean Value Theorem, there is a number c in (0, 2) such that

$$f(2) - f(0) = f'(c)(2 - 0)$$

Now f (2) =6, f (0) =0, and $f'(x) = 3x^2 - 1$, so this equation becomes

6= $(3c^2 - 1)$2=$6c^2 - 2$

which gives $c^2 = \frac{4}{3}$, that is c=$\pm 2/\sqrt{3}$. But c must lie in (0, 2), so c=$2/\sqrt{3}$.

Example: Suppose that f (0) =-3, and $f'(x) \leq 5$ for all values of x. How large can possibly be?

Solution: We are given that f is differentiable (and therefore continuous) everywhere. In particular, we can apply the Mean Value Theorem on the interval [0, 2]. There exists a number c such that

$$f(2) - f(0) = f'(c)(2 - 0)$$
$$f(2) = \text{f}(0) + 2f'(c) = -3 + 2f'(c)$$

We are given that $f'(x) \leq 5$ for all x, so in particular we know that $f'(c) \leq 5$. Multiplying both sides of this inequality by 2, we have $2f'(x) \leq 10$, so

$$f(2) = -3 + 2f'(c) \leq -3 + 10 = 7$$

The largest possible value for is 7.

Practice:

1. Using the fundamental theorem of calculus, explain why taking the integral of $f(x) = x^{-3/2}$ over [-1,1] cannot be solved using the integration method defined in calc 1.

2. Evaluate the definite integrals using the Fundamental Theorem of Calculus.

   a. $\int 3x^2 dx$ [0,5]

   b. $\int (x^2 - 7x + 12)\, dx$ [0,3]

Solutions:

1. The function $f(x) = x^{-\frac{3}{2}}$ is not continuous on the interval [-1,1] which violates the fundamental theorem of calculus.

2. Integration questions

   a. 125

   b. 27/2