Reviewing Trigonometry

Introduction and Definition

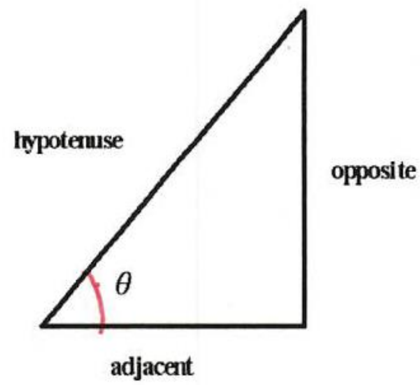
Students often think they are unprepared for Calculus if they are not fully comfortable with trigonometry. However, trigonometry is not in any way the material that precedes a course in Calculus. In a Calculus course, trigonometric functions are studied for the properties they share with other familiar functions such as polynomial and rational functions, rather than for their special properties or applications.

In short, students should know the values of the trigonometric functions at certain angles using radian measure instead of degrees ($\sin\frac{\pi}{6}=\frac{1}{2}$ rather than $\sin 30° = \frac{1}{2}$). The easiest way to do this is by using the *unit circle*. Students should also be familiar with the graphs of trigonometric function.

Trigonometry is first learned using right triangles. Given an acute angle θ in a right triangle, the definitions for the trigonometric functions are:

$\cos\theta = \frac{\text{adj}}{\text{hyp}}$ $\quad \sin\theta = \frac{\text{opp}}{\text{hyp}}$ $\quad \tan\theta = \frac{\text{opp}}{\text{adj}}$

$\sec\theta = \frac{\text{hyp}}{\text{adj}}$ $\quad \csc\theta = \frac{\text{hyp}}{\text{opp}}$ $\quad \cot\theta = \frac{\text{adj}}{\text{opp}}$



There are certain special angles for which we can find the values of the trigonometric functions. The values for $\sec\theta, \csc\theta,$ and $\cot\theta$ are found by computing the reciprocals of $\cos\theta, \sin\theta,$ and $\tan\theta$.

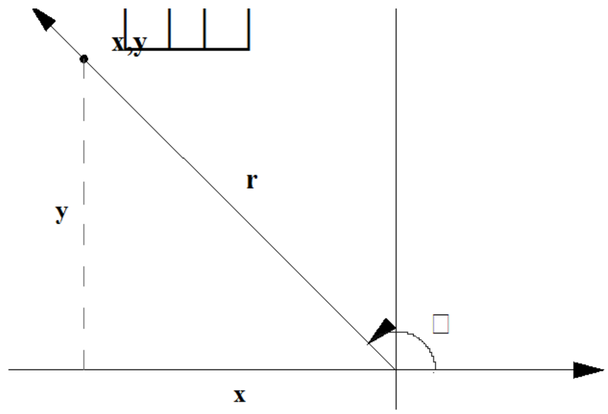
θ (degrees)	θ (radians)	$\cos\theta$	$\sin\theta$	$\tan\theta$
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$

Degrees are converted to radians by multiplying by $\frac{\pi}{180°}$. To convert from radians to degrees, multiply by $\frac{180°}{\pi}$. For example, $30°\cdot\frac{\pi}{180°}=\frac{\pi}{6}$.

In Precalculus, the definition is extended to any angle. An angle is said to be in standard position if its vertex is at the origin and the initial side is along the positive x-axis. Counterclockwise is considered the positive direction.

Model Problems:

Example 1: Let (x, y) be a point on the terminal side of an angle θ, and $r = \left(\sqrt{x^2 + y^2}\right)$ the distance of (x, y) to the origin: Represent the basic trig definitions.



The definitions are as follows:

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\sec\theta = \frac{r}{x} \qquad \csc\theta = \frac{\mathrm{r}}{\mathrm{y}} \qquad \cot\theta = \frac{x}{y}$$

If you look at the acute angle made by the terminal side and the x-axis (in this case the negative part of the x-axis) the definitions are the same as for a right triangle.

Note that every trigonometric function can be defined using $\cos\theta$ and $\sin\theta$.

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

The Pythagorean identities follow from the above definition:

$$\sin^2\theta + \cos^2\theta = 1 \qquad (x^2 + y^2 = r^2 \text{ implies } \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1),$$

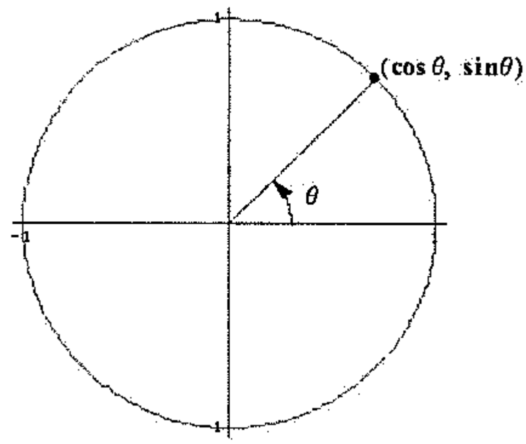
$\tan^2\theta + 1 = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$. (The last two identities can be obtained from the first by dividing the entire equation by $\cos^2\theta$ or $\sin^2\theta$).

Notation

Trigonometric functions are written differently than other functions you are familiar with that involve combinations of arithmetic expressions in x, such as $f(x)=2x^2+2x-1$. Here you can simply plug in the value for x to find y. The trigonometric functions, instead, are given names like $\sin x$ or $\cos x$. One common mistake is to think that $\cos 2x$ is equal to $2\cos x$. However, $\cos 2x$ is the composition of $f(x)=\cos x$ with $g(x)=2x$, that is $f(2x)$, which is different from $2f(x)=2\cos x$. For example, $1=2\cos\frac{\pi}{3}\neq\cos 2\cdot\frac{\pi}{3}=-\frac{1}{2}$. You should also avoid writing expressions such as $\sin=\frac{1}{2}$. There always has to be an angle or an expression after sin. For instance, $\sin\frac{5\pi}{6}=\frac{1}{2}$.

The Unit Circle

The most useful way to remember values of the trigonometric functions is by using the unit circle. where $r=1$, so the definitions for $\cos\theta$ and $\sin\theta$ reduce to $\cos\theta=x$ and $\sin\theta=$ y. That is, any point (x,y) on the unit circle can be written as $(\cos\theta,\sin\theta)$. To emphasize this, $\cos\theta$ is the x-coordinate of the point, and $\sin\theta$ is the y-coordinate of the point on the unit circle. See the diagram below.



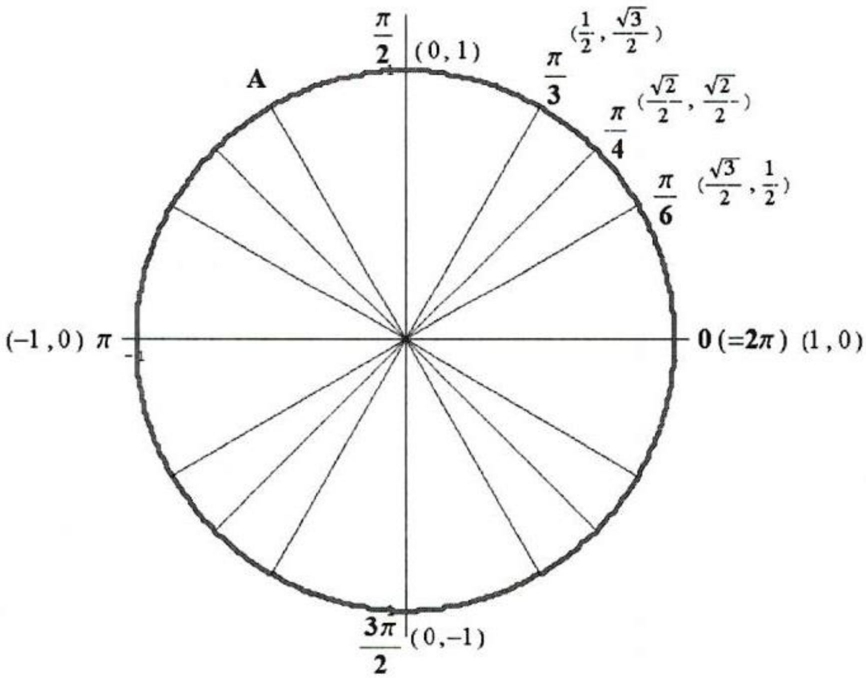
If the independent variable x is used instead of θ, then $(\cos x,\sin x)$ represents the coordinates of the point on the unit circle. This can initially be confusing having the function $\cos x$ as the x-coordinate.

Example 2: Let (x,y) be a point on the unit circle on the terminal side of the angle θ. Represent the basic trig functions.

$$\cos\theta=x \qquad \sin\theta=y \qquad \tan\theta=\frac{y}{x}$$

$$\sec\theta=\frac{1}{x} \qquad \csc\theta=\frac{1}{y} \qquad \cot\theta=\frac{x}{y}$$

The unit circle can be used to display known values of the trigonometric functions. So far, we have the following:



So, for example, $\sec\frac{\pi}{3} = \frac{1}{\frac{1}{2}} = 2$. We can also find the values of the trigonometric functions at integer multiples of $\frac{\pi}{2}$ $(90°)$. For example, $\sin\frac{3\pi}{2} = -1$. The other angles that need to be filled in are integer multiples of $\frac{\pi}{6}$ (30°)and $\frac{\pi}{4}$ (45°). The values of these trig functions can be determined from their corresponding acute angles on the unit circle. These acute angles are called "reference angles". By definition, a reference angle is the acute angle formed both by the terminal side of the angle and the horizontal axis. The correct signs of the coordinates are determined according to which quadrant the point is in.

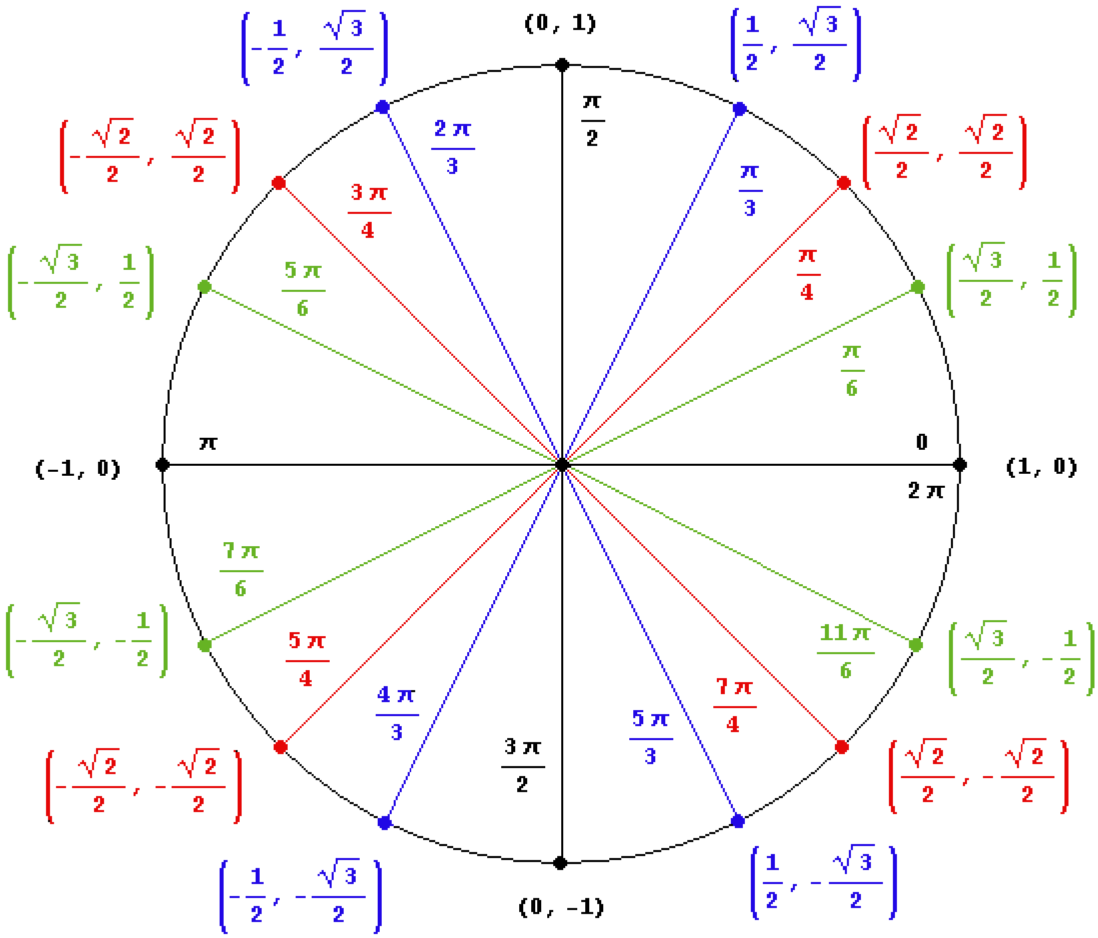
The reference angle of an angle θ is the difference (always taken to be positive) between the terminal side of θ and the horizontal axis. For angle in Quadrants II and III, this is the difference with the negative part of the x-axis, and for angle in Quadrant IV, it is the difference with the positive x-axis.

Example 3: Find the angle and coordinates on the unit circle that correspond to the terminal side labeled A in the diagram above.

The terminal side is in Quadrant II, so the reference angle is $\frac{\pi}{3}$. The angle is then $\pi - \frac{\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3}$.

The y-coordinate is the same as for $\frac{\pi}{3}$, but the x-coordinate is negative. The point on the unit circle is therefore $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$.

Example 4: Fill in the rest of the unit circle given above.

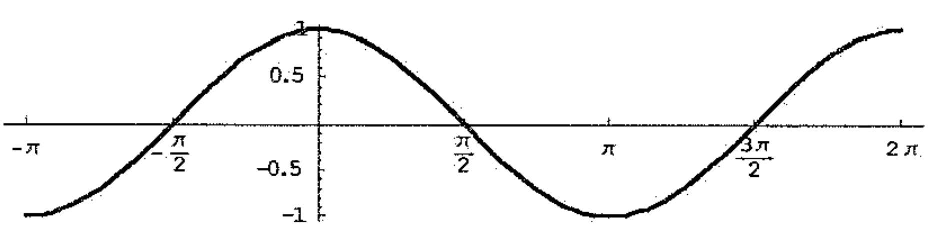


Notice that if 2π is added to or subtracted from any angle θ, the terminal side of the angle is the same. Therefore, the value of any of the trigonometric functions is unchanged if you add or subtract an integer multiple of 2π to the given angle θ.

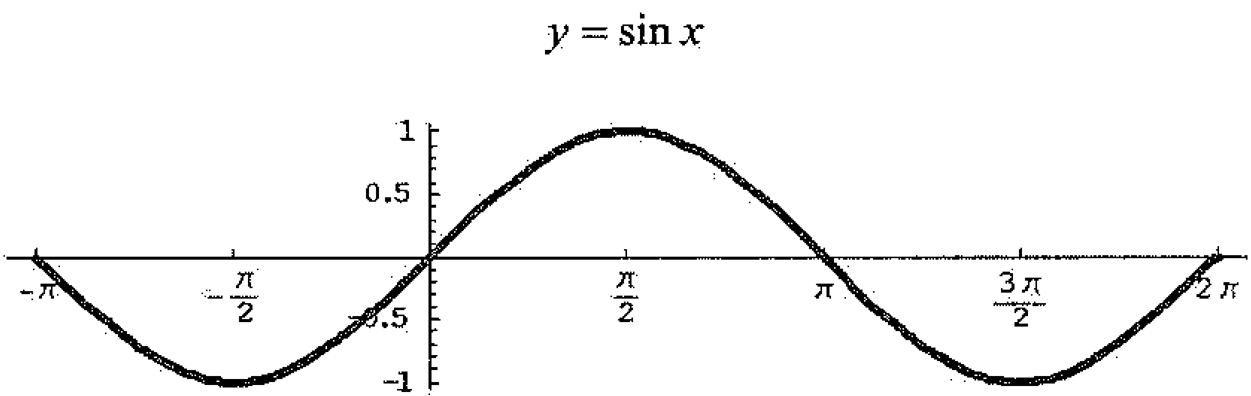
Graphs of the trigonometric functions

Below are the graphs of the 6 trigonometric functions. All have period 2π (which means that the graph repeat itself in intervals of 2π), except for $y = \tan x$ and $y = \cot x$ which have period π.

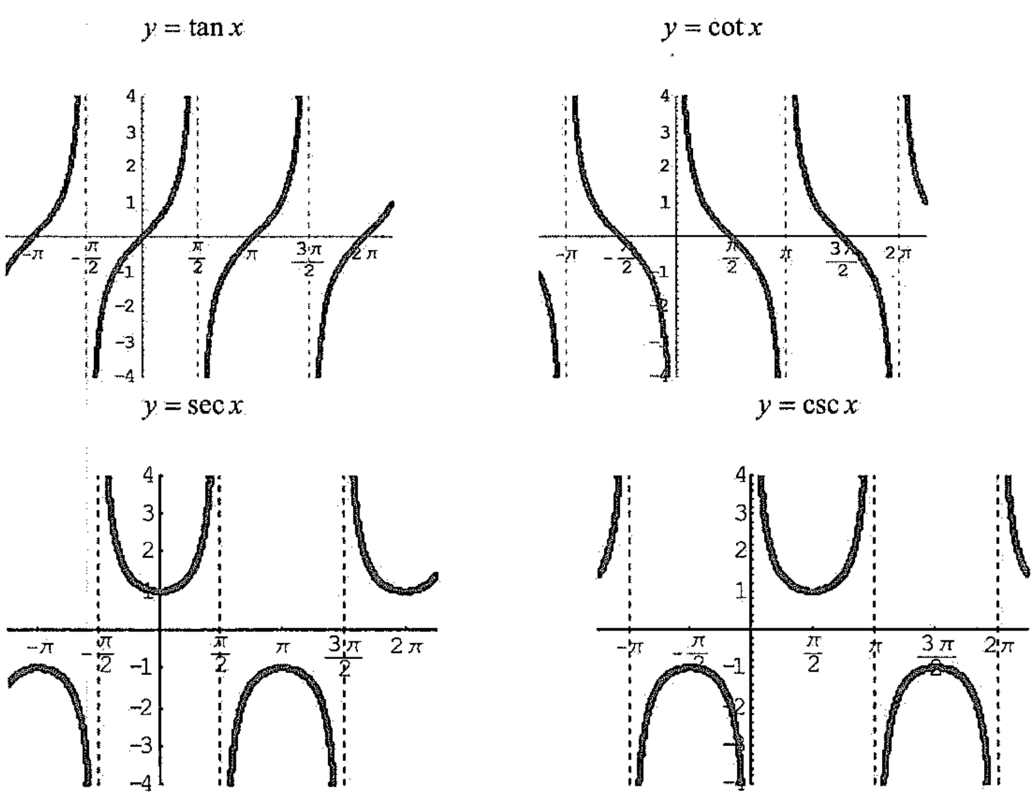
$y = \cos x$



The function $y = \cos x$ is an example of an even function. Even functions satisfy $f(-x) = f(x)$. Graphically, even functions are symmetric with respect to the y-axis.



The function $y = \sin x$ is an example of an odd function. Odd functions satisfy $f(-x) = -f(x)$. Graphically, odd functions are symmetric with respect to the origin. The function $y = \tan x$ and $y = \cot x$, are also odd functions.



Practice Exercises:

1. Complete the unit circle without looking at the completed one in the lesson.

2. Find the following values.

a) $\tan\frac{3\pi}{4}$ b) $\sec\frac{7\pi}{6}$ c) $\csc\frac{3\pi}{2}$ d) $\cot(-\frac{\pi}{3})$

3. Find the following values. (Hint: you can add or subtract multiples of 2π to obtain an angle between 0 and 2π that has the same terminal ray as the given angle).

a) $\cos\frac{21\pi}{4}$ b) $\sin(-\frac{32\pi}{3})$

Answers:

2. a) -1 b) $\frac{-2}{\sqrt{3}} = \frac{-2\sqrt{3}}{3}$ c) -1 d) $-\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

3. a) $-\frac{\sqrt{2}}{2}$ b) $-\frac{\sqrt{3}}{2}$