

12. The Tangential Component of Newton's 2nd Law:

A. Static Equilibrium and Torque

B. Angular Momentum Conservation

C. Center-of-Mass Calculations

Introduction and Summary

A. Thus far in this course we have mostly dealt with the motion of point-like masses and we have ignored the fact that most objects have some spatial extent. This not only simplified our calculation but it also is a reasonably good first approximation. However, for some kinds of problems the size of the object cannot be neglected. For example, if an object has an extent or size, the translational motion is important and additionally the rotational motion might need to be considered.

B. The last module treated angular motion and it was emphasized that it is useful to write Newton's 2nd law for these problems as having a radial component and a tangential component. Mostly the last module was involved with the radial component because the angular velocity was taken as constant.

C. When a tangential force is applied to an object in angular motion, then the object changes its angular velocity and the object accelerates. The tangential component of Newton's 2nd law can be written in terms of torque (instead of force), angular acceleration (instead of tangential acceleration), and moment of inertia (instead of mass).

D. You might expect that there are conservation laws for angular motion and there are:

Conservation of Angular Kinetic Energy plus Linear Energy and Conservation of Angular Momentum. Sometimes it is your choice whether to use Newton's laws in angular form or the conservation laws. An example of a problem you probably would want to use the Conservation Laws would be a sphere rolling down an irregularly shaped hill. If on the other hand, the hill were the hypotenuse of a triangle, then you could use either Newton's Laws or the Conservation Laws it would be your choice.



The Concept of Center of Mass:

An extended object can be replaced (for some purposes) by an imaginary point-like object

1. having the same mass as the extended object and
2. located at the center of mass

All of the mass of an extended object is assumed to be at the center of mass for the purposes of studying the effect of external forces on the object.

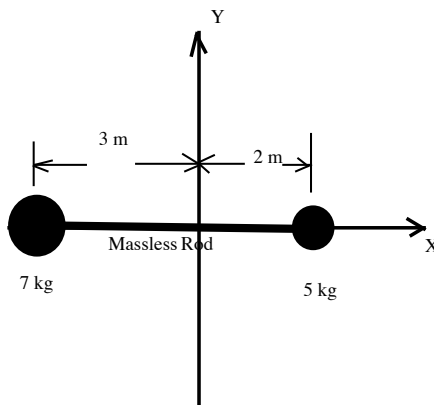
If you are interested in the angular motion of an object, then you must keep in mind that the object has a size or extent and simply replacing the object with the center of mass is not enough.



Determination of the Center-of-Mass Location

Probably the simplest way to understand how to calculate the center of mass (COM) is to do some examples for systems of point masses. While the masses that make up the system are point objects, the masses are connected by a rigid (but weightless) rod to form an extended object. The COM is a weighted average of the positions of the point masses.

EXAMPLE #1: Suppose two point masses m_1 and m_2 are attached to a rigid rod that is parallel to the x-axis so they move together as a system. For simplicity, suppose the rod has no mass. For example:



The COM \bar{X} is calculated using the formula

$$\bar{X} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \quad \text{or} \quad M \bar{X} = m_1 x_1 + m_2 x_2$$

where $M \equiv m_1 + m_2$, the total mass. The first expression above will be used to calculate \bar{X} , the COM mostly because it allows you to calculate the COM \bar{X} . The second expression emphasizes that it is the weighted average of the two masses which is replaced by an object having the total mass M and located at the COM. Substitution of the numerical values for this example you get

$$\bar{X} = \frac{5 \text{ kg} \times 2 \text{ m} + 7 \text{ kg} \times (-3 \text{ m})}{5 \text{ kg} + 7 \text{ kg}} = -0.91 \text{ m}$$

since using *Mathematica* yields

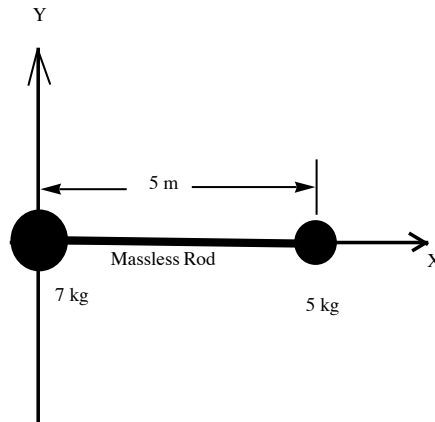
$$\frac{5 * 2 + 7 * (-3.)}{5 + 7}$$

-0.916667

So the COM of this system has a mass of 12 kg and is located on the minus X axis a distance of -0.91 m from the origin of coordinates. Another way of saying this is that the COM is (3 m - 0.91 m) = 2.1 m to the right of the 7 kg mass.



EXAMPLE #2: Suppose you have the same system as in EXAMPLE #1 but suppose the coordinate system you choose to use has its origin at the location of one of the masses as in the diagram below:



You apply the formula for calculating the COM and your first idea is that since the $m_2 = 7 \text{ kg}$ is at the origin of coordinates, this mass will not contribute anything to the COM calculation. But your first guess is wrong because m_2 is required in the calculation of the total mass M so m_2 has an effect on the COM calculation even though it is at the origin of coordinates. Proceeding with the calculation

$$\bar{X} = \frac{5 \text{ kg} \times 5 \text{ m} + 7 \text{ kg} \times (0 \text{ m})}{5 \text{ kg} + 7 \text{ kg}} = \frac{25}{12} \text{ m} = 2.08 \text{ m}$$

since *Mathematica* gives

25 / 12 .

2.08333

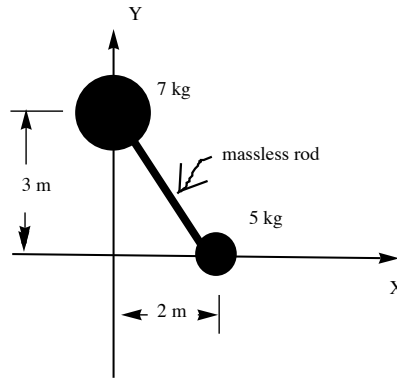
Notice this answer is **effectively the same** as when we used the original coordinates in EXAMPLE #1 because \bar{X} is 2.08 m to the right of the 7 kg object in both cases.

EXAMPLE #3: The COM of a system with three Masses m_1 , m_2 , and m_3 located along the x axis at x_1 , x_2 , and x_3 respectively is obtained by a simple generalization of the formula used in EXAMPLE #1 for two masses, that is

$$\bar{X} = \frac{m_1 X_1 + m_2 X_2 + m_3 X_3}{m_1 + m_2 + m_3}$$



EXAMPLE #4: Suppose you want to calculate the COM for a system of two masses connected by a massless rod and the two masses are NOT along the x-axis but instead are arranged as below:



This COM calculation requires two equations:

$$\bar{X} = \frac{m_1 X_1 + m_2 X_2}{m_1 + m_2} \quad \text{and} \quad \bar{Y} = \frac{m_1 Y_1 + m_2 Y_2}{m_1 + m_2}$$

Application to the problem at hand yields

$$\bar{X} = \frac{5 \text{ kg} \times 2 \text{ m} + 7 \text{ kg} \times 0 \text{ m}}{5 \text{ kg} + 7 \text{ kg}} = \frac{17}{12} \text{ m} = 1.4 \text{ m} \quad \text{and} \quad \bar{Y} = \frac{5 \text{ kg} \times 0 \text{ m} + 7 \text{ kg} \times 3 \text{ m}}{5 \text{ kg} + 7 \text{ kg}} = \frac{21}{12} \text{ m} = 1.75 \text{ m}$$

since

$$17. / 12$$

$$1.41667$$

$$21 / 12.$$

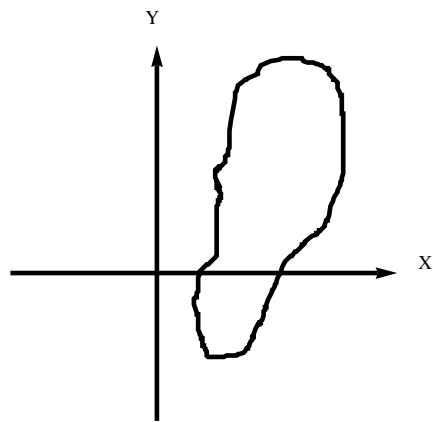
$$1.75$$

So in this case the COM is not on either the x-axis or y-axis but instead is at $\bar{X} = 1.4 \text{ m}$ and

$\bar{Y} = 1.75 \text{ m}$.

Determination of the COM of an Irregularly Shaped Object

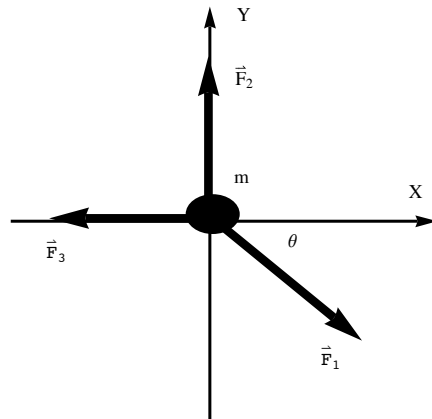
The center of mass of an irregular object like that indicated below is handled by dividing the object up into small bits of mass and then calculating the COM by treating the point-like objects using the method above. (If the object is fairly simple, the methods of calculus can be used to calculate the COM; however, we do not do this in this course.)



When the whole object is broken up into smaller pieces in order to calculate the COM, the small pieces need not be all the same size or shape. What is necessary is the mass of each small bit and the location of the bit relative the coordinate system.

The First Condition for Equilibrium

Suppose three forces \vec{F}_1 , \vec{F}_2 , and \vec{F}_3 act on a mass m as indicated by the diagram below. The forces may be thought of as acting through ropes attached to the mass.



For the mass m not to move (that is, for it to be in "equilibrium"), the net force on it must be zero:

$$\sum \vec{F} = 0 \quad (\text{the 1st Condition of Equilibrium})$$

and this is shorthand for $\sum F_x = 0$ and $\sum F_y = 0$. This follows from Newton's 2nd Law. For the forces in the example above we get for the x-components of force

$$F_1 \cos[\theta] - F_3 = 0 \quad \text{or} \quad F_1 \cos[\theta] = F_3$$

and for the y-components of force we get

$$F_2 - F_1 \sin[\theta] = 0 \quad \text{or} \quad F_1 \sin[\theta] = F_2$$

A NUMERICAL EXAMPLE: Suppose $F_2=30$ Nt and $F_3 = 25$ Nt. Determine F_1 and θ so that the mass is in equilibrium.

ANSWER: $F_1 \cos[\theta] = F_3 = 25$ Nt and $F_1 \sin[\theta] = F_2 = 30$ Nt. So we have two equations for the two unknowns F_1 and θ . Dividing the second equation by the first equation we get

$$\frac{F_1 \sin[\theta]}{F_1 \cos[\theta]} = \frac{F_2}{F_3} \quad \text{and after cancellation of the } F_1 \quad \text{and using } \tan[\theta] = \frac{\sin[\theta]}{\cos[\theta]} \quad \text{we get}$$

$$\tan[\theta] = F_2 / F_3 = 30 \text{ Nt} / (25 \text{ Nt}) = 6/5$$

$\theta = \text{ArcTan}[6./5]$ which *Mathematica* gives as $\theta = 0.88$ Rad or $\theta = 50.2^\circ$.

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ArcTan[6 / 5.]
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```
0.876058
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```
ArcTan[6 / 5.] * (180 / π)
```

```
50.1944
```

We can now solve for F_1 since from the first equation $F_1 = F_3 / \cos[\theta] = 25 \text{ Nt} / \cos[50.2^\circ] = 38.9$ Nt

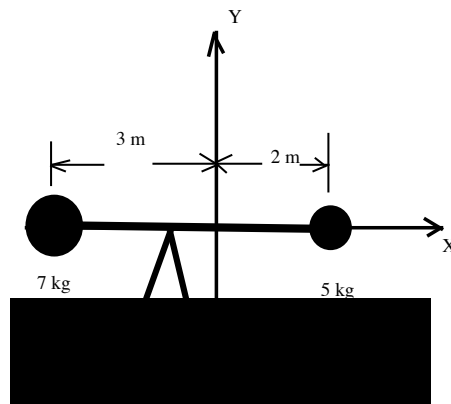
```
25. / Cos[50.]
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38.89309567151031`
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◀ | ▶

The 2nd Condition for Equilibrium

Since the sum of the forces equal to zero is the 1st Condition you would guess there is a 2nd Condition and you would be right. The 2nd Condition is necessary for objects that have extension or a size. You see that there is a need for a 2nd condition by returning to the 1st example of two masses connected by a mass-less rod. We know the COM is about 1.0 m to the left of the origin of coordinates. Suppose you change the system by placing the mass-less rod on a fulcrum or pivot located just under the COM and this pivot is sitting on the ground.



*Aside: You might recognize this as a model of a children's teeter-totter where one child is heavier than the other child. You might also remember that for the teeter-totter to **balance** (which is another term for **equilibrium**), the heavier child must be closer to the pivot point than the lighter child.*

1st Condition for Equilibrium: $\sum \mathbf{F} = 0$

EXAMPLE: There are three forces acting on the mass-less rod: Two forces downward due to the force of gravity on the 7 kg ($68.2 \approx 70$ Nt.) mass and 5 kg (≈ 49 Nt.) mass and also one force upward from the pivot on the mass-less rod. The 1st condition of equilibrium tells us the upward force by the pivot is $68.2 \text{ Nt} + 49 \text{ Nt} = 117.2 \text{ Nt}$. This is true regardless of where the pivot is located.

But this is not enough to have the system in equilibrium. Unless the pivot is placed at the COM the system will not be in equilibrium. The system will have a tendency to rotate either clockwise or counter clockwise depending upon the location of the pivot. It is clear that another condition for equilibrium is needed. This condition involves **torque** τ .

Definition of Torque τ

$$\text{Torque} = \mathbf{Force} \times \mathbf{moment\ arm}$$

The concept of force is clear but the concept of moment arm (or lever arm) has not been discussed yet. Maybe the best way to understand the concept of **moment arm** is with some examples. But before that, here is the statement of the second Condition for equilibrium.

The 2nd Condition for Equilibrium

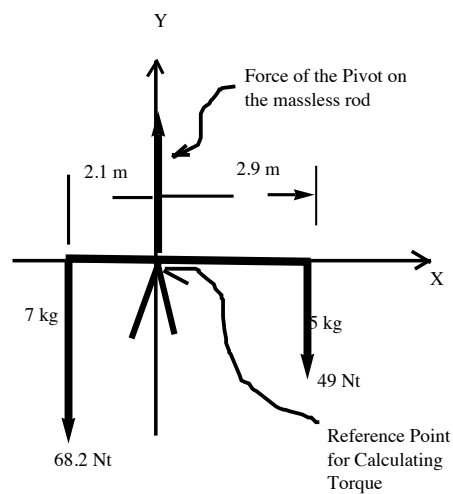
$\sum \tau = 0$ The sum of the torques τ must be zero in order for an object with extent (that is, size) to be in equilibrium.



The Reference Point for Calculating Torques

The **reference point** is the origin of coordinates for calculating moment arms and torque. The choice of the reference point is arbitrary, just as the choice of a frame of reference is arbitrary. You can choose the reference point to make the problem as simple as possible to solve.

EXAMPLE: Going back to the teeter-totter example, the diagram below is a sort of free-body diagram for this example.



The Lever Arm or Moment Arm

The lever arm is the shortest distance from the line of force to the reference point.

EXAMPLE: There are three forces acting on the mass-less rod and we calculate the moment arm (or lever arm) and torque for each force.

1. The 68.2 Nt force has a moment arm of about 2.1 m. The torque τ_1 associated with this force is

$$\tau_1 = 68.2 \text{ Nt} \times 2.1 \text{ m} = 143.2 \text{ Nt} \times \text{m}$$

This torque would cause a counter-clockwise of the mass-less rod if it were the only torque acting so it is considered a positive torque (by convention).

2. The 49 Nt force has a moment arm of about 2.9 m. The torque associated with this force is

$$\tau_2 = 49 \text{ Nt} \times 2.9 \text{ m} = -142.1 \text{ Nt} \times \text{m}$$

3. The 117.2 Nt force of the pivot on the mass-less rod produces a torque τ_3 which is

$$\tau_3 = 117.2 \text{ Nt} \times 0 \text{ m} = 0$$

since the line of force goes through the reference point the moment arm is zero m.

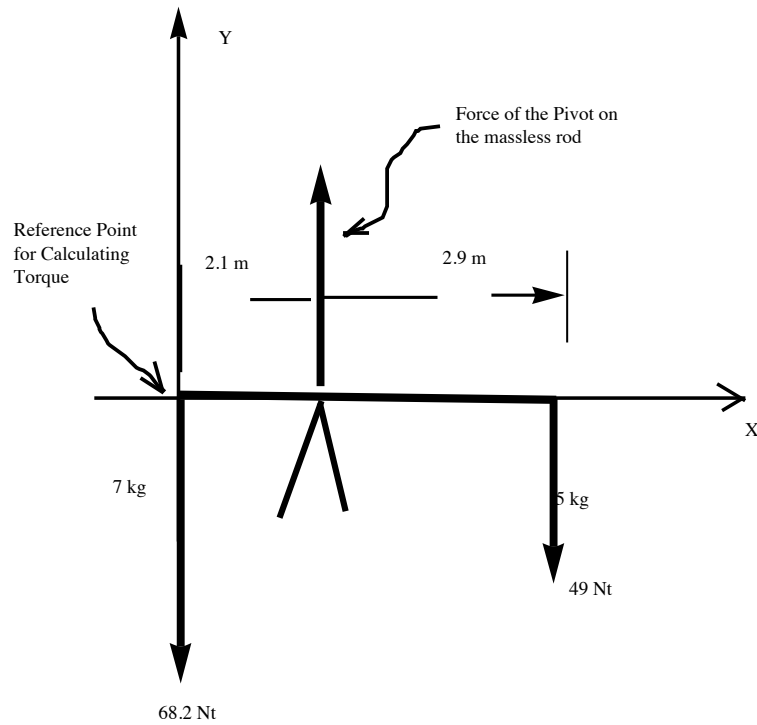
The Sum of the Torques:

$$\tau_1 + \tau_2 + \tau_3 = 143.2 \text{ Nt} \times \text{m} - 142.1 \text{ Nt} \times \text{m} = 1.1 \text{ Nt} \times \text{m} \approx 0$$

Thus the 2nd condition of equilibrium is satisfied to within roundoff error. This provides a justification for putting the pivot where it is in order to have the teeter-totter balance.

The Choice of the Reference Point is Arbitrary

Let us look at this problem with the reference point at a different location and see how this affects the calculation of the torques and the 2nd condition of equilibrium. Move the reference point to the 7 kg mass so we have the situation below:



The torque τ_1 of the 68.2 Nt force is zero in this case since the moment arm is zero. The line of force passes through the reference point so the shortest distance from the reference point to the line of force is zero.

$$\tau_1 = 68.2 \text{ Nt} \times 0 \text{ m} = 0 \text{ Nt} \times \text{m}$$

The torque τ_2 of the 117.2 Nt force of the pivot on the mass-less rod is given by

$$\tau_2 = 117.2 \text{ Nt} \times 2.1 \text{ m} = + 246.1 \text{ Nt} \times \text{m}$$

and this torque is consider positive since it would produce a counter-clockwise rotation of the mass-less rod. The torque τ_3 due to the 49 Nt force is

$$\tau_3 = 49 \text{ Nt} \times 5 \text{ m} = - 245 \text{ Nt} \times \text{m}$$

The sum of the torque is in this case is zero to within roundoff error

$$\tau_1 + \tau_2 + \tau_3 = 0 \text{ Nt} \times \text{m} + 246.1 \text{ Nt} \times \text{m} - 245 \text{ Nt} \times \text{m} \simeq 0$$

CONCLUSION: The 2nd condition of equilibrium says that the sum of the torques is independent of the reference point chosen to solve the problem. Therefore you can choose the reference point to make solving the problem as easy as possible.



The Angular Form of Newton's 2nd Law:

Recall that Newton's 2nd law for angular problems consists of two parts:

The radial part: $F_R = m a_R$

The tangential part: $F_T = m a_T$

1. The radial part of Newton's 2nd law includes the radial acceleration $a_R = v^2 / r$ where v is the tangential velocity and r is the radius of the angular motion relative the axis of rotation. The radial force F_R is a real force like gravity that holds the Earth in its orbit around the Sun or the electric force that holds atoms together. The radial piece was studied in the last chapter and in the chapter on gravitation.

2. The tangential part of Newton's 2nd law is quite often written using $a_T = \alpha r$ in the form

$F_R = m \alpha r$. Then multiply both sides of the equation by r obtaining after a slight rearrangement

$$(F_T r) = m(\alpha r) r = m r^2 \alpha$$

The tangential force F_T is applied to the object which is a radial distance r from the axis of rotation so the torque τ is given by $\tau = F_T r$ and this appears on the left hand side of the above equation.

The moment of inertia I is defined $I = m r^2$ so the tangential part of Newton's 2nd law becomes

$$\tau = I \alpha .$$