13. Oscillatory or Periodic Motion

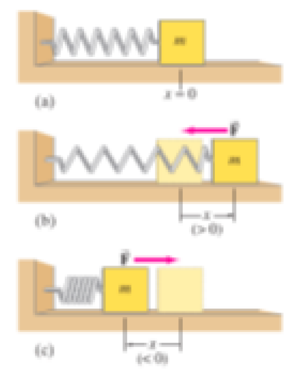
Summary and Outline

"Oscillatory motion" is motion that repeats over and over again after a time T that is called the "period." This motion is also called "periodic motion" with a "repeat time" T. Examples of periodic motion include (1) a mass on a spring, (2) the simple pendulum, (3) the motion of a planet like the Earth about the Sun, and (4) the rotational motion of a disk like a music CD. Mostly the focus will be on the mass & spring system and a simple formula for the period of the motion will be obtained. Also conservation of energy will be applied to this system.

◄ | ►

The Mass m and Spring System

A simple example of this is pictured below



NOTES:

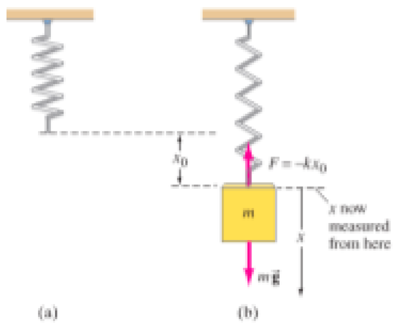
1. The mass is attached by a spring to the wall and (for simplicity) the mass slides on the floor without resistance.
2. There is an equilibrium position of the mass where it can be placed and it will not move in the future (unless the mass is displaced from equilibrium) because there is no horizontal force on the mass.
3. The horizontal x axis has x=0 at the center of mass of the block of wood.
4. If the mass is moved to the right of the equilibrium position the spring stretches and produces a force in the negative x direction.
5. The law of springs or Hooke's law says that the force due to the spring is proportional to the displacement x from the equilibrium point:

$$F = - k x$$

6. The proportionality constant k is called the "spring constant" and the minus sign is there because when x is positive, the spring force is in the negative x direction.

How the Spring Constant k Is Measured

Imagine the spring is hung from the ceiling and at first there is no mass attached to the bottom end of the spring.



1. Then add a m=1 kg to the bottom end of the spring and suppose for the spring stretches x = 0.5 m
2. Next add another 1 kg mass (for a total mass of 2 kg) to the bottom of the spring and the spring stretches an additional 0.5 m until the CM of the block is at x=1.0 m
3. Below is some further data taken:

m	x
0 kg	0 m
1 kg	0.5 m
2 kg	1.0 m
3 kg	1.5 m

The mass in the first column is multiplied by g=9.8 m/s^2 so the mass results in a gravitational force via mg. Because the gravitational force is in equilibrium with the upward force of the spring, we also know the spring force. The data table below has the second column x the same as the above table but the first column is the spring force F=mg:

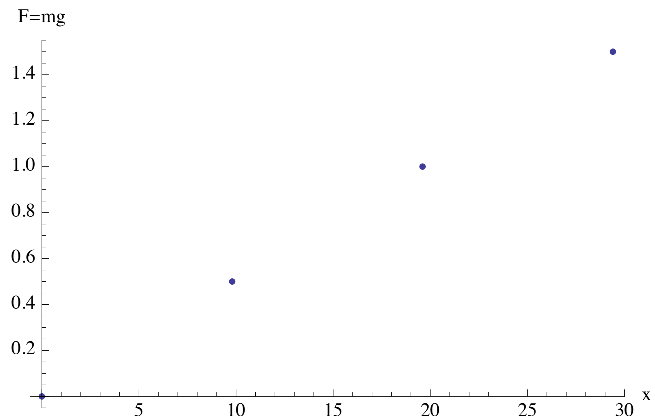
data =

0	0
9.8	0.5
19.6	1.0
29.4	1.5

;

```
ListPlot[data, AxesLabel → {"x", "F=mg"}]
```

•



The four data points fall on a straight line and the slope of that line is the spring constant k. The spring constant is calculated from

$$k = \frac{\Delta F}{\Delta x}$$

where any two points can be used. For example, using points 2 and 3 we get

$$k = \frac{19.6\text{ Nt} - 9.8\text{ Nt}}{1.0\,m - 0.5\,m} = \frac{9.8}{0.5} = 19.6\text{ Nt/m}$$

You should check to see that you get the same k if you use any two points.

Stiffness of the Spring: A spring is stiffer than another spring if for the same additional mass, the stiffer spring stretches less far. Convince yourself that the spring constant k of the stiffer spring is larger.

Newton's 2nd Law Applied to the Mass and Spring System

Newton's 2nd law F = m a is combined with the law of springs F = - k x to get

$$m\,a = -k\,x$$

Since the acceleration a=Δv/Δt the above equation of motion can also be written

$$\frac{\Delta V}{\Delta t} + \frac{k}{m}x = 0$$

There are two unknowns in this equation: the position x[t] of the end of the spring and the velocity v[t] of the mass. But the definition of velocity v

$$V = \frac{\Delta X}{\Delta t}$$

gives a 2nd equation which, when combined with the equation above, can be used to solve the problem.

The Most Important Quantity of Periodic Motion:

The period T or repeat time is the most important thing about the mass and spring system. The formula for T requires a little calculus but the result obtained is fairly simple and states that the period T of the motion is given by

$$T = 2\pi\sqrt{\frac{m}{k}}$$

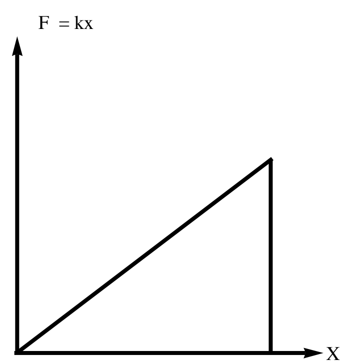
Dimensional Analysis: You might guess the answer for the period by noting there are only two parameters in this problem: the mass m in kg and the spring constant k in Nt/m. The period T must be constructed by combining m and k in some way to get a time T, and the simplest way is what you try first. Recall a Newton = kg×m/sec^2 so the units of k are also kg/sec^2. One way (the simplest) of combining m and k to get a time is to first divide m/k and you get something with units of sec^2. o what we should do is take the square root of m/k to get a quantity having units of time in seconds. Thus our guess is that T=$\sqrt{m/k}$ based on simple dimensional analysis. Of course we didn't get the 2π but we got something in the ballpark. Another way of getting T will be given later.

The Potential Energy of a Spring

The work done on a spring equals the change in potential energy of the spring:

$$W = -\Delta PE$$

This is the definition of the change in potential energy. It clearly takes work ΔW to compress a spring by Δx and that work is $\Delta W = F_{\text{spring}}\,\Delta x = -k\,x\,\Delta x$. The spring force is variable and depends upon the x you start with. We did the example of the work done by a variable before and it was decided that the work done is equal to the area under the force F versus x graph:



The area is just a triangle which is 1/2 the base (which is x) times the height F=-kx so W=-(1/2)(kx)x or more simply

$$W = -\frac{1}{2} k\, x^2$$

Combining this with $W = -\Delta PE$ we get for a spring

$$\Delta PE = \frac{1}{2} k\, x^2$$

The Total Energy of the Mass and Spring System

The total energy is the kinetic energy plus potential energy, thus

$$\text{TE} = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$$

The maximum potential is where x=A, the amplitude of the motion. The potential is then

$\text{PE}_{\text{max}} = \frac{1}{2} k A^2$ but the kinetic energy there is zero since v=0 when x=A.

The maximum kinetic energy is where x=0 at the equilibrium position. $\text{KE}_{\text{max}} = \frac{1}{2} m v_{\text{max}}^2$.

Conservation of energy tells us that

$$\text{PE}_{\text{max}} = \text{KE}_{\text{max}}$$

or in other words

$$\frac{1}{2} k A^2 = \frac{1}{2} m v_{\text{max}}^2$$

Solving for the maximum velocity we get

$$v_{\text{max}} = \sqrt{\frac{k}{m}} A$$

which is a simple relation between the amplitude A and the maximum velocity. Next we will use this to get the period T of the motion.

◄ | ►

The Period of Oscillation for the Mass on a Spring Problem

We know of two kinds of periodic motion:

1. The mass on a spring problem which has a period T
2. The motion of a point on the circumference of a circle of radius R

Key Idea: The two kinds of motion are similar because they are periodic. We know of only function that is periodic and that is the Cosine function, so we write

$$X[t] = A\ Cos[\theta] \quad \text{where } \theta=\omega t$$

The angular position θ[t] depends upon time for a circle having a constant angular velocity ω.

Combining we get

$$X[t] = A\ Cos[\omega t]$$

Note that when t=0 the Cos[0]=1 and X[0]=A, which is good since the oscillation of the mass is started at t=0 by pulling the mass a distance A from the equilibrium position. (The Sine function is periodic too but using X[t] = A Sin[ωt] does not work since when t=0 the Sine function is zero, which does not provide the correct solution for X[0]=A.)

The **analogy** between circular motion and the oscillatory motion of a mass on a spring can be taken further by associating the

(1) "effective radius" A (previously called R) of the circular motion with the amplitude A and

(2) the tangential speed with v_{max}

(**See demonstration referenced below with the spray paint which drives home the analogy between circular motion and the mass on a spring problem**.)

The period T of the oscillatory motion can be computed using s=vt where s is the distance traveled in time t of an object moving with velocity v. The motion is circular so the distance traveled is along the circumference in a time t=T for one complete revolution and appropriate velocity is the tangential velocity v which is constant. Taking the time t=T (the period) and s=2πA (the circumference of a circle having radius A) we get the connection

$$2\pi A = v_{max} T$$

v_{max} is the maximum velocity of the mass on the spring, which is when it passes through the equilibrium position. (The velocity of the mass is zero when X=A and the velocity of the mass is at its maximum when X=0, which is the equilibrium position.) Solving for the period T we get

$$T = \frac{2\pi A}{v_{max}}$$

But before we got $v_{max} = \sqrt{\frac{k}{m}}\, A$ so substitution yields the formula for the period of a mass on a spring problem

$$\boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

Stiffer springs have larger spring constants k since for the same force the displacement is smaller. Since k is in the denominator of the formula for the period T above, this means that stiffer springs have shorter periods of oscillation than less stiff springs. Also, if the mass attached to the spring is increased (while keeping k the spring constant the same) then the period T increases. See Prof. Lewin's demonstration referenced below to see this effect.

Demonstration Reference:

Lecture #10 Harmonic Motion
Walter Lewin MIT Open Courseware Initiative

Introductory Physics 1st Semester Homepage (This is a calculus-based course but the demonstrations are worthwhile for PHYS 110 students.)
http://ocw.mit.edu/OcwWeb/Physics/8-01Physics-IFall1999/CourseHome/

Lecture #10 Harmonic Motion Homepage where you download the lecture in iTunesU MP4 format, RM format, regular MP4, etc.
http://ocw.mit.edu/OcwWeb/Physics/8-01Physics-IFall1999/VideoLectures/detail/embed10.htm

See in particular the mass & spring demonstration. A paint can is attached to the oscillating mass and the spray paint comes out on a piece of paper moving with a constant speed in front of the spray can opening. This demonstrates the connection between up and down oscillatory motion and the Cosine function where the horizontal axis is time and the vertical axis is displacement X of the mass. This connection is important in obtaining the formula for the period T of the mass & spring system.

There are also some demonstrations of the mass & spring system with numerical values for the mass, spring constant, and observed period.

Also, see the pendulum demonstration for a particular length pendulum and observed period of motion.