Lecture #3. Kinematic Relations

Introduction and Summary

A. The previous lecture introduced you to the quantities used to describe motion:

(1) position and displacement, (2) velocity and speed, and (3) acceleration.

B. The concept of average velocity was compared with the concept of instantaneous velocity.

C. Most of this course will deal with problems where the acceleration is constant.

The mass on a spring and the pendulum are two cases where acceleration is not constant.

D. This lecture will focus on the connection between the quantities used to describe motion: kinematic relations

E. There are a total of five kinematic relations (two of which were introduced in the last lecture.

F. These five kinematic relations are important in predicting the motion of an object when the acceleration is constant

and these will be referred to as the 5 Magic Equations

The First Kinematic Relation:

The Velocity: Recall that if you collect data of positions x of a moving object at successive times t AND you graph the position x versus time t, THEN it is natural to focus on the slope of the graph. The slope of the x versus t graph in physics is called the velocity.

TWO CASES:

1. Linear Graph: if x versus t is a straight line, then the velocity is constant.

2. Non-linear Graph: if x versus t is not a straight line, then the velocity is not constant and there is acceleration.

In this case

a) the slope of the line between two points on the graph is called the average velocity $\langle v \rangle$ which is given by

$$\langle v \rangle = \frac{x_2 - x_1}{\Delta t} \tag{1}$$

The notation in equation (1) is a little different from the previous lecture but the meaning is the same. x_1 is the position of the object at time t_1 and x_2 is the position of the object at time t_2 and $\Delta t = t_2 - t_1$ is the time between the two position measurements.

You should probably try to remember equation (1) written in the following equivalent form

$$x_2 = x_1 + \langle v \rangle \Delta t$$

b) the slope of the tangent to the graph at point 1 is called the instantaneous velocity v. v is obtained by imagining the second observation point is closer and closer to the first observation point. This means that $t_2 \to t_1$ in equation (1) above or equivalently $\Delta t \to 0$

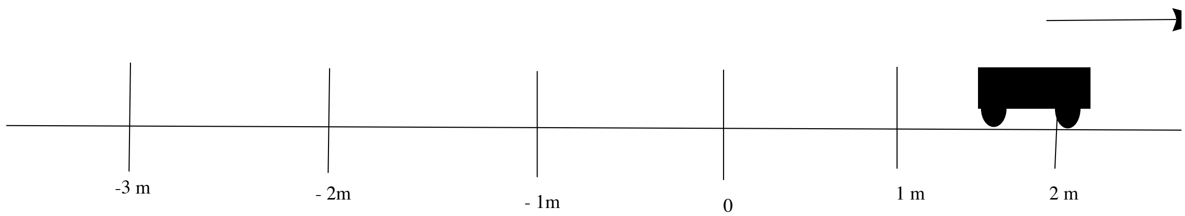
$$v = \frac{x_2 - x_1}{\Delta t} \text{ when } \Delta t \to 0$$

Equation (1) is one of the 5 magic equations for kinematics.

AN EXAMPLE OF THE USE OF EQUATION 1:

Prediction of the Location of an Object at Some Time in the Future.

Recall the example from the previous lecture of an object moving to the right



When position x was graphed versus time t wet got a straight line graph with slope $\langle v \rangle$ = + 2 m/s.

(The average velocity is a constant in this case since the graph of x versus t is a straight line.)

Suppose at t_1 = 0 Sec the object is at the origin of coordinates so x_1 = 0 meters.

QUESTION: Where will the object be when t_2 = 7.5 Sec? Using equation (1) we get

$$x_2 = x_1 + \langle v \rangle \Delta t = 0\,\text{m} + 2\,\frac{\text{m}}{\text{s}} \times 7.5\,\text{s} = 15\,\text{m}$$

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0 + 2 * 7.5
15.
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So the prediction is that the object will be at position x_2 = 7.5 m when Δt=7.5 s has elapsed. Note this is a prediction of

what will happen in the future. Not only is this a later time, but the time is at a half second interval and last lecture

we imagined taking data only at one second intervals.

The Second Kinematic Equation

1. Remember that when the graph of position x versus time t is not a straight line, the slope of the tangent to the graph
at a point is called the instantaneous velocity v. The slope of the tangent line is changing with time, so it follows that
the instantaneous velocity v is changing.

2. The instantaneous velocity v can be graphed versus time t and the slope of this graph is called the acceleration a.
The instantaneous acceleration a will be constant for most of this course and for this case, the instantaneous acceleration a
is equal to the average acceleration ⟨a⟩ which is constant.

The average acceleration ⟨a⟩ is defined for two points on the velocity v versus time t graph as

$$\langle a \rangle = \frac{v_2 - v_1}{\Delta t} \tag{2}$$

The notation in equation (2) is a little different from the previous lecture but the meaning is the same. v_1 is the velocity of the object at time t_1 and v_2 is the velocity of the object at time t_2 and $\Delta t = t_2 - t_1$ is the time between the two velocity measurements. You should probably try to remember equation (2) written in the following equivalent form

$$v_2 = v_1 + \langle a \rangle \Delta t$$

Equation (2) is the second kinematic equation that will be used to solve problems of motion.

AN EXAMPLE OF THE USE OF EQUATION 2:

Prediction of the Instantaneous Velocity v of an Object at Some Time in the Future.

Recall the example from the previous lecture of data collected from observing a falling object acted upon by gravity. The average acceleration was obtained from the data as $\langle a \rangle = 9.8\ m/s^2$. Suppose at time $t_1 = 0\ s$ you release an object from rest which means that $v_1 = 0\ m/s$. What is the velocity v_2 of the object after $t_2 = 4.5$ s have elapsed so $\Delta t = (t_2 - t_1) = (4.5\ s\ 0\ s) = 4.5$ s?
(What this really means is "what is the velocity at 4.5 Sec?")

The answer is obtained from equation (2) and thus

$$v_2 = v_1 + \langle a \rangle \Delta t = 0\ m/s + 9.8\ m/s^2 \times (4.5\ s - 0\ s) = 44.1\ m/s$$

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0 + 9.8 * (4.5 - 0)
44.1
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So the velocity of the object is 44.1 m/s after a time 4.5 s has elapsed.

Unit Cancellation:

Notice the seconds units in time t canceled one of the seconds units in the m/s^2 to give the units of velocity m/s. Units are useful
in checking to see if your calculation is correct. The units in all terms of the equation should agree with each other.

The Third Kinematic Equation:

There is an additional equation for the average velocity ⟨v⟩ which is different from equation (1) and this new equation only applied in the case where the acceleration is constant (which is true most of the time in this course). Suppose the velocity changes from v_1 to v_2 over the time Δt then the average velocity ⟨v⟩ can be calculated using

$$\langle v \rangle = \frac{v_1 + v_2}{2} \tag{3}$$

which is a really simple formula and it is not very hard to prove. If the acceleration is constant then the slope of the velocity versus time graph is a straight line and we use this fact in the proof of equation (3). But first let us try out equation (3) in a specific case to see if it indeed works.

Example: Suppose we use the same data given in the last lecture for the falling object (the first column is

the elapsed time and the second column are the corresponding velocities.)

vdata =

0	0
1	9.8
2	19.6
3	29.4
4	39.2
5	49
6	58.8

;

```
ListPlot[vdata, PlotStyle → PointSize[0.02], AxesLabel → {"time (sec)", "velocity (m/s)"}]
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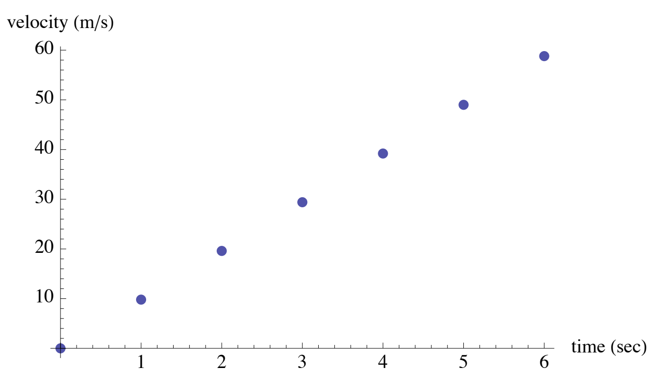

It is reasonable to interpolate the data between the data points so we get

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ListPlot[vdata, PlotStyle → PointSize[0.02],
 AxesLabel → {"time (sec)", "velocity (m/s)"}, PlotJoined -> True]
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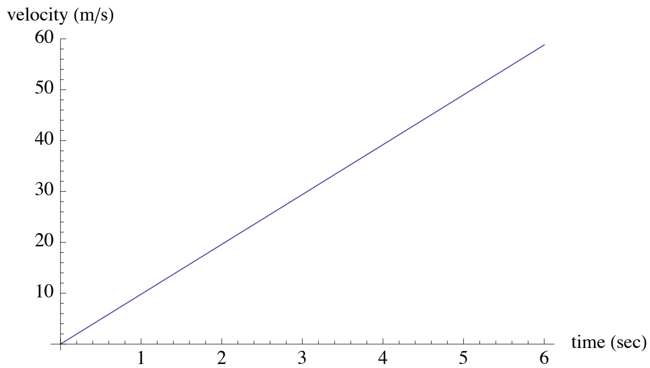


Again to be concrete suppose we are interested in the average ⟨v⟩ from $t_1 = 2$ sec where

$v_1 = 19.6\, m/s$ to $t_2 = 5$ sec where $v_2 = 49\, m/s$.

Using equation (3) we get the average velocity as

$$\langle v \rangle = \frac{19.6 + 49}{2}$$

34.3

This is the same numerical value as obtained from equation (1) since we also happen to have the position versus time data

time (sec)	0	1	2	3	4	5	6
Position y (meters)	0	4.9	19.6	44.1	78.4	122.5	176.4

so when $t_1 = 2$ sec where $y_1 = 19.6\, m$ and $t_2 = 5$ sec where $y_2 = 122.5\, m$. Applying equation (1) to y motion you get

$$\langle v \rangle = \frac{y_2 - y_1}{\Delta t}$$

$$\langle v \rangle = \frac{122.5 - 19.6}{5 - 2}$$

34.3

Perhaps remarkably we got the same numerical answer for ⟨v⟩=34.3 m/s. In this case, we have

position y versus time data but you might

NOT have this data available and yet you might like to be able to calculate the average velocity.

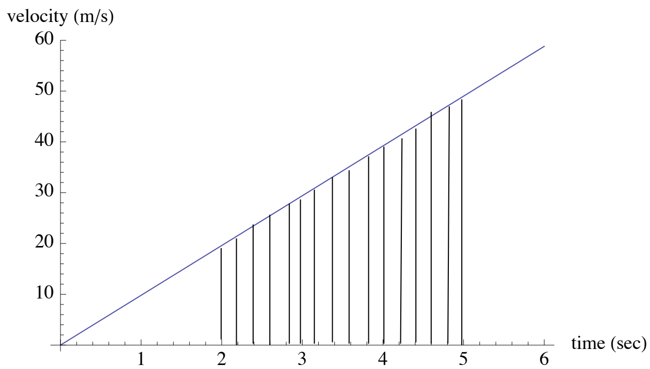
Proof of the Third Kinematic Equation:

You can skip this proof on the first reading.

Take the velocity versus time graph above and divide the interval from $t_1 = 2$ sec to $t_2 = 5$ sec

into N=30 sub-intervals

each having a width 0.2 seconds thus



More generally the interval $\Delta t = t_2 - t_1$ can be divided into N sub-interval each having width h given by

$$h = \frac{t_2 - t_1}{N} \quad \text{or} \quad N = \frac{\Delta t}{h}$$

(By the way, the larger N, the more accurate the calculation below becomes.)

The time of each sub-interval is $t_1 + n \times h$ where n=0, 1, 2, ..., N and $t_1 + N \times h = t_2$ which you

can get from the equation

just above. The velocity at each sub-interval is $v[t_1 + n\,h]$. The average velocity $\langle v \rangle$ is then

$$\langle v \rangle = \frac{v[t_1] + v[t_1 + h] + v[t_1 + 2\,h] + v[t_1 + 3\,h] + \ldots + v[t_1 + N\,h]}{N}$$

Aside: If you want to calculate the average height of a person in this classroom then you would add up all the person's heights

and divide by the number of people. A similar argument gives the average velocity in the above equation. You add up the velocities of each sub-interval and then divide by the number N of sub-intervals. But using N= Δt/h the above equation becomes

$$\langle v \rangle = \frac{v[t_1] + v[t_1 + h] + v[t_1 + 2\,h] + v[t_1 + 3\,h] + \ldots + v[t_1 + N\,h]}{(\Delta t \,/\, h)}$$

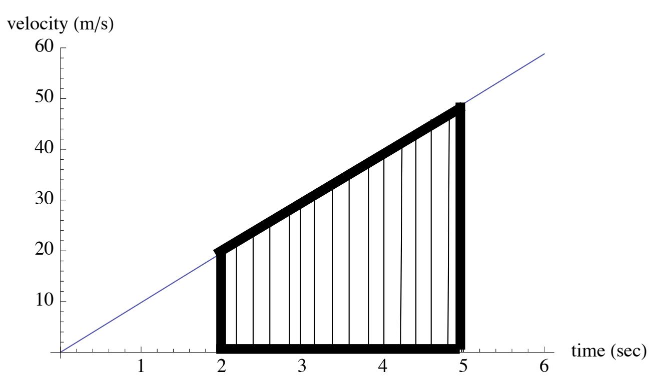
or after a little algebra

$$\langle v \rangle = \frac{(v[t_1] + v[t_1 + h] + v[t_1 + 2\,h] + v[t_1 + 3\,h] + \ldots + v[t_1 + N\,h]) * h}{\Delta t \,/\, h)}$$

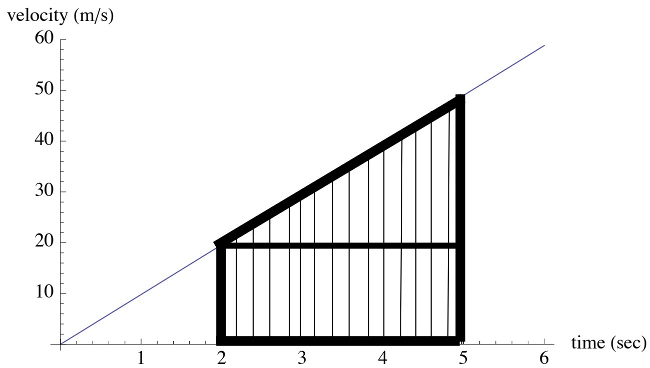
and one more step yields

$$\langle v \rangle = \frac{v[t_1] * h + v[t_1 + h] * h + v[t_1 + 2\,h] * h + v[t_1 + 3\,h] * h + \ldots + v[t_1 + N\,h] * h}{\Delta t}$$

The secret of the third kinematic equation is to interpret the NUMERATOR of above equation as the AREA indicated below bounded by the dark lines:



The area indicated above is just the AREA of a rectangle having area = $\Delta t \times v_1$ PLUS the area of a triangle = $\frac{1}{2} \times \Delta t \times (v_2 - v_1)$



The sum of the area of the triangle plus rectangle is

$$\text{Total Area} = \Delta t \times v_1 + \frac{1}{2} \times \Delta t \times (v_2 - v_1) = \frac{\Delta t}{2} \times (v_1 + v_2)$$

Using the Total Area in the numerator of the equation for $\langle v \rangle$ yields

$$\langle v \rangle = \frac{\frac{\Delta t}{2} \times (v_1 + v_2)}{\Delta t}$$

The Δt in the numerator and the Δt denominator cancel and this equation reduces the third kinematic equation

$$\langle v \rangle = \frac{v_1 + v_2}{2}$$

The Fourth Kinematic Equation

Another kind of prediction problem you might want to solve is the following. Suppose the falling mass acted on by gravity

is at the origin of coordinates initially. That is $y_1 = 0$ meters at $t_1 = 0$ Sec. The object is then released from rest that is $v_1 = 0$ m/s.

How far will the object fall in Δt=8.5 seconds?

By the way, it is known that objects fall under the force of gravity always with an acceleration of a=9.8 m/s^2 on Earth if you

neglect air resistance. This is true regardless of the objects mass. Since this acceleration a on Earth is so common in physics

it is given a special name g=9.8 m/s^2.

The Longer Solution to this Problem: We know the initial velocity $v_1 = 0$ m/s , the acceleration a=9.8 m/s^2, and the time of travel

Δt=8.5 Sec, equation #2 allows us to get the velocity v_2 at $t_2 = 8.5$ Sec via

$$v_2 = v_1 + \langle a \rangle \Delta t = 0 + 9.8\, m/s * 8.5\, s$$

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0 + 9.8 * 8.5
83.3
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So the velocity is $v_2 = 83.3$ m/s when Δt=8.5 Sec. *Note that $\Delta t=(t_2-t_1) = t_2$ since t_1=0.*

However, we still do not know how far the mass has fallen. However, we can calculate the average velocity using

equation #3 that is

$$\langle v \rangle = \frac{v_1 + v_2}{2} = \frac{0 + 83.3\, m/s}{2} = 41.65\, m/s$$

$$\frac{0 + 83.3}{2}$$

41.65

Now with equation (1) we may final out how far the mass has fallen

$$y_2 = y_1 + \langle v \rangle \times \Delta t = 0\,\text{m} + 41.65\,\frac{\text{m}}{\text{s}} \times 8.5\,\text{s} = 354\,\text{m}$$

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0 + 41.65 * 8.5
354.025
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So the prediction is the object will have fallen 354 m after 8.5 Sec.

This solution is a bit round about but there is a more direct way to proceed.

More Direct Solution to this Problem: The fourth kinematic equation is

$$y_2 = y_1 + v_1\,\Delta t + \frac{1}{2}\,a\,t^2 \tag{4}$$

This is has not yet been derived but let us use it first to see its value. By the way, often the motion is in the x direction

but here it is in the y direction.

We know that $y_1 = 0$ meters at $t_1 = 0$ Sec and the object is released from rest (that is $v_1 = 0$ m/s) and also

the time of fall is Δt=8.5 seconds. Also since we are dealing with gravity a=g=9.8 m/s^2. Using this information in kinematic equation #4 we get

$$y_2 = 0\,\text{m} + (0\,\text{m}/\text{s} \times 8.5\,\text{sec}) + \frac{1}{2}\,9.8\,\text{m}\big/\text{s}^2 * (8.5\,\text{s})^2$$

$$y_2 = (0 * 8.5) + (1/2) * 9.8 * 8.5^2$$

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354.025
```

So equation #4 predicts the mass will have fallen 354 m which is the same answer as obtained by the roundabout method.

Proof the Fourth Kinematic Equation

The proof is quite easy and we will do it for motion in the x direction. Basically we repeat the steps used above in the
round about method. But if you do these steps once and for all, you can save time in problem solving.

Start with equation #1 written a

$$x_2 = x_1 + \langle v \rangle \Delta t$$

Use the kinematic equation #3 in the form

$$\langle v \rangle = \frac{v_1 + v_2}{2}$$

to eliminate the average velocity $\langle v \rangle$ obtaining

$$x_2 = x_1 + \left(\frac{v_1 + v_2}{2}\right) \Delta t$$

Next use equation #2 to eliminate

$$v_2 = v_1 + \langle a \rangle \Delta t$$

to eliminate v_2 and obtain

$$x_2 = x_1 + \left(\frac{v_1 + v_1 + \langle a \rangle \Delta t}{2}\right) \Delta t = x_1 + \left(\frac{2 v_1 + \langle a \rangle \Delta t}{2}\right) \Delta t$$

and after a very little amount of algebra we finally get

$$x_2 = x_1 + v_1 \Delta t + \frac{1}{2} \langle a \rangle \Delta t^2 \qquad (5)$$

which is the Fourth Kinematic Equation. This equation can be used to predict where an object is at a given time
if you know the initial position, the initial velocity of the object as well at the average acceleration.

The Fifth Kinematic Equation

One additional equation can be obtained from equation (1) with $\Delta x = x_2 - x_1$ written as

$$\Delta x = \langle v \rangle \Delta t$$

and using equation (3) for the average velocity

$$\langle v \rangle = \frac{v_1 + v_2}{2}$$

and from equation (2) you can solve for the Δt

$$\Delta t = \frac{v_2 - v_1}{\langle a \rangle}$$

After substitution we get

$$\Delta x = \left(\frac{v_1 + v_2}{2}\right)\left(\frac{v_2 - v_1}{\langle a \rangle}\right)$$

and with a little rearrangement (algebra)

$$2 \langle a \rangle \Delta x = (v_1 + v_2)(v_2 - v_1)$$

The binomial on the right hand side can be simplified by multiplied and after a cancellation you get

the fifth kinematic equation

$$2 \langle a \rangle \Delta x = v_2{}^2 - v_1{}^2$$

which is usually written in the form

$$v_2{}^2 = v_1{}^2 + 2 \langle a \rangle \Delta x$$

An Example: Using the 5th Kinematic Equation

Suppose you have a falling object and $\langle a \rangle$= 9.8 m/s^2 that you release from rest so $v_1 = \; m/s$ and

you want to know how fast the object is falling after a distance Δx=7 m. Solve the 5th kinematic

equation

for v_2 and write

$$v_2{}^2 = v_1{}^2 + 2 \langle a \rangle \Delta x$$

and given what we know this becomes

$$v_2{}^2 = (0 \, m/s)^2 + 2 \times 9.8 \, m/s \times 7 \, m$$

$$\mathbf{v_2 = \sqrt{2 * 9.8 * 7}}$$

11.7132

So the object is moving at 11.7 m/s when it has fallen a distance Δx=7 m.

Galileo's Leaning Tower of Pisa Experiment on the Moon by Astronaut David Scott

Historical Reference

Some doubt that Galileo ever dropped anything off the tower itself. However, a close examination of his writings reveals that he described the effects that can only be found if he performed the experiments himself. He described that it was almost impossible to release a heavy and a light ball simultaneously, that he always released the light ball first. Then, as the heavy object fell a little faster, it would catch up and eventually both will land at the same time. The structure of Pisa tower is such that one needs to lean forward to drop something. It turns out that muscle fatigue makes one release a light object first. . Basically, you tend to release the lighter object first. Galileo was aware of this and also has some idea that air resistance might be playing a role. He observed that the lighter ball always seemed to get a tiny head start, but that the heavier ball "caught up" to the lighter one. The reference: I.Bernard Cohen "The birth of a new physics".

David Scott performs Galileo's experiment by dropping a hammer and a feather on the Moon in a video at the web location below. Note the hammer and feather both fall much slower than you might expect for an object dropped on Earth. You can past it into your web browser. But before doing that, try clicking on the hyperlink below in blue which should take you to NASA. The longer version of the
David Scott video is the recommended one as it has some out takes in it.

http://nssdc.gsfc.nasa.gov/planetary/lunar/apollo_15_feather _drop.html