Lecture #4: Vector Addition

Background and Introduction

i) Some physical quantities in nature are specified by only one number and are called scalar quantities. An example of a scalar quantity is temperature, and another is mass.

ii) Other physical quantities in nature are specified by two or more numbers taken together. An example, is the displacement in two dimensions which requires the **magnitude** as well as the **direction** of the motion. Force and velocity are two other examples of vector quantities. Mostly in this lecture the displacement vector will be used as the example of a vector.

iii) It is possible to write the laws of nature without using vectors (and this was done until the late 1800's). However, the laws of nature are more simply written and understandable using vector notation.

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Definitions:

1. Thus far we have discussed motion only in one dimension because

   a) motion in one dimension occurs in nature and

   b) one dimensional problems are simpler to understand.

2. The concept of displacement in one dimension has some aspects of vectors.

   The displacement is defined $\Delta x = x_2 - x_1$

   a) The size or **magnitude** of Δx is given by $|\Delta x|$.

   Example $|\Delta x| = 5$ meters means you could either

   walk 5 meters in the $+ x$ direction or 5 meters in the $- x$ direction.
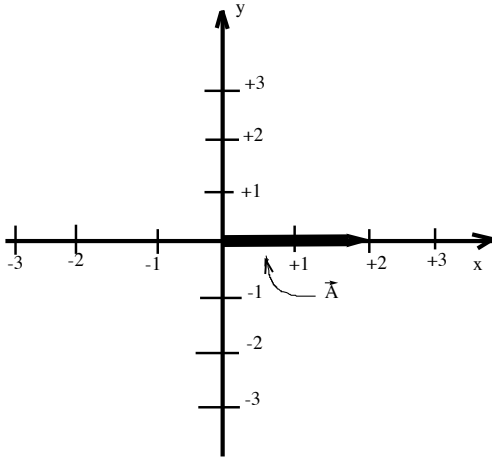
   b) The **sign** of Δx is also important. Example $\Delta x = -5$ meter

The vector concept is a generalization of these concepts to two dimensions and three dimensions.
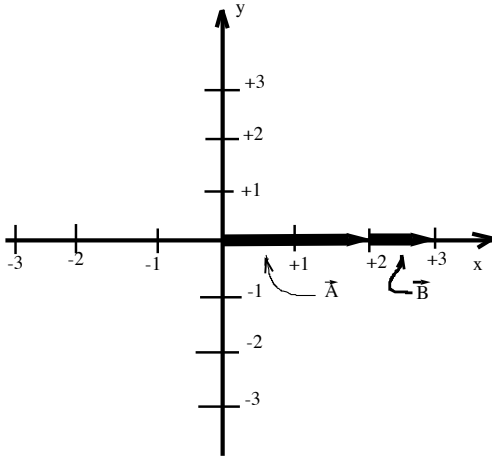
Addition of Displacement Vectors in One Dimension

Suppose at first we have a displacement in the + x-direction of 2 m away from the origin as
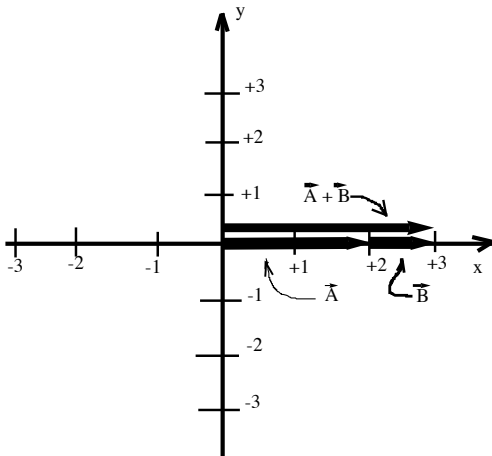
indicated below by the vector symbol $\vec{A}$

Following the first displacement, suppose a second displacement $\vec{B}$ of magnitude + 1 m is also in the positive x direction.

This is clearly equivalent to a single displacement of magnitude 3 m in the + x direction from the origin and is written as $\vec{A} + \vec{B}$
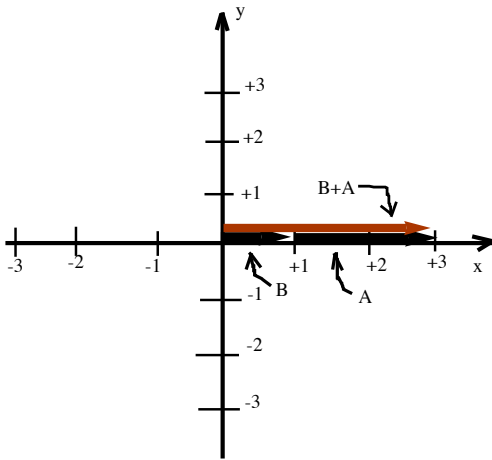
So the sum of two displacements $\vec{A} + \vec{B}$ is itself a displacement.

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The Order of the Displacements.

Doing the $\vec{B}$ displacement first and following that with the $\vec{A}$ displacement is written $\vec{B} + \vec{A}$.

You can convince yourself $\vec{B} + \vec{A} = \vec{A} + \vec{B}$ because you wind up in the same spot. The order of the vector addition is not important.
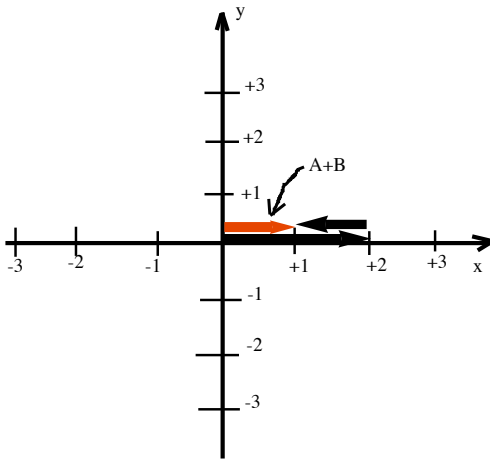
The Effect of the Sign of the Displacement

What happens if one of the displacements is negative?

Example $\vec{B}$= -1m which a displacement in the negative x direction and $\vec{A}$ the same.

Answer: $\vec{A} + \vec{B} = (2\text{-}1)\text{m}= 1$ m. Graphically you have in this case
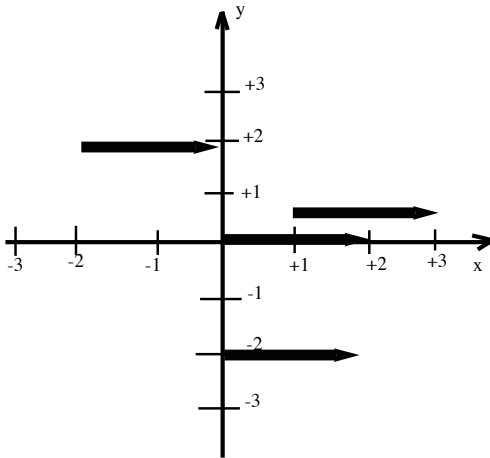

So $\vec{A} + \vec{B}$ in this case has its tail at the origin and tip at +1 meters.

Equal Vectors:

The magnitude and sign (direction) of the displacement are all that is important in defining a vector.

All the vectors below are considered equal because they all have length 2 meter and are in the
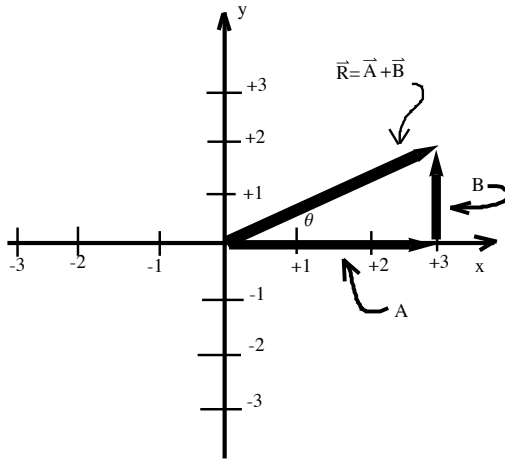
positive x direction.

The Graphical Procedure for Addition of Two Vectors:

a) You add vector $\vec{A}$ to vector $\vec{B}$ by moving vector $\vec{B}$ parallel to itself until the "tail" of $\vec{B}$ is at the tip

of the vector $\vec{A}$. The sum of the two vectors $\vec{A} + \vec{B}$ is drawn from the "tail" of $\vec{A}$ to the tip of $\vec{B}$.

b) Alternatively, you can add vector $\vec{A}$ to vector $\vec{B}$ by moving vector $\vec{A}$ parallel to itself until the "tail"

of $\vec{A}$ is at the tip of the vector $\vec{B}$. The sum of the two vectors $\vec{B} + \vec{A}$ is drawn from the "tail" of $\vec{B}$ to

the tip of $\vec{A}$.

c) It should be clear that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ and this holds in two and three dimensions as well.

d) General Property of Vectors: If you have a vector $\vec{A}$ having a given magnitude and direction,

then you can move vector $\vec{A}$ anywhere and as long as the magnitude and direction are the same,

you have the same vector $\vec{A}$.

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Addition of Two Vectors NOT in the Same Direction:

Suppose the vector $\vec{A}$ has length 3m in the positive x-direction and the vector $\vec{B}$ has length 2 and

is in the positive y-direction. The sum of the two vectors $\vec{A}+\vec{B}$ is indicated in the diagram below:

Notice the Following:

1. The sum of two vectors $\vec{A}$ and $\vec{B}$ is itself a vector since you can walk directly from the origin to the tip of $\vec{B}$. The sum of two vectors is called the **Resultant** Vector $\vec{R}=\vec{A}+\vec{B}$. Since $\vec{R}$ is a vector, its magnitude $|\vec{R}|$ and direction θ are specified (see item #4 below).

2. **Graphical Method** for Adding Vectors: Add vector $\vec{A}$ to vector $\vec{B}$ by moving vector $\vec{B}$ parallel to itself until the "tail" of $\vec{B}$ is at the tip of the vector $\vec{A}$. The sum of the two vectors $\vec{A} + \vec{B}$ is drawn from the "tail" of the first vector $\vec{A}$ to the tip of the second vector $\vec{B}$.

3. The order of addition of $\vec{A}$ and $\vec{B}$ can be reversed and you will get the same Resultant Vector $\vec{R}=\vec{B}+\vec{A}$. So the order two vectors are added is unimportant $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

4. The length or magnitude of the Resultant Vector $|\vec{R}|$ is computed using the Pythagorean Theorem that is $|\vec{R}|=\sqrt{(3\ m)^2 + (2\ m)^2}$. Using *Mathematica* we compute the numerical value in this case

$$\sqrt{3^2 + 2.^2}$$

$$3.60555$$

So the length of the Resultant Vector is $|\vec{R}|$=3.6 m.

3. The direction θ of the Resultant Vector is the angle θ with respect to the x direction indicated in the diagram above. The angle θ can be computed using the Tangent function or Tan for short and in this case $\text{Tan}[\theta] = (y\ /\ x) = \frac{2\ m}{3\ m}$=0.67 The angle θ is obtained from the ArcTan function since θ=ArcTan[(y/x)]=ArcTan[0.67]=33.7° since y=3 m and x= 2m in the example. If you have *Mathematica* do the calculation you get

$$\text{ArcTan}[2\ /\ 3.] * \left(\frac{360.}{2\ \pi}\right)$$

$$33.6901$$
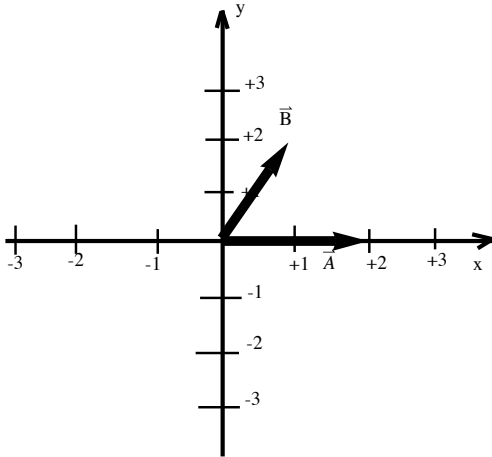
so the angle $\theta \cong 34°$. The factor $(360/2\pi)$ converts from angular measure in radians to degrees.

The are 360° in a circle an equivalently there are 2π Radians.
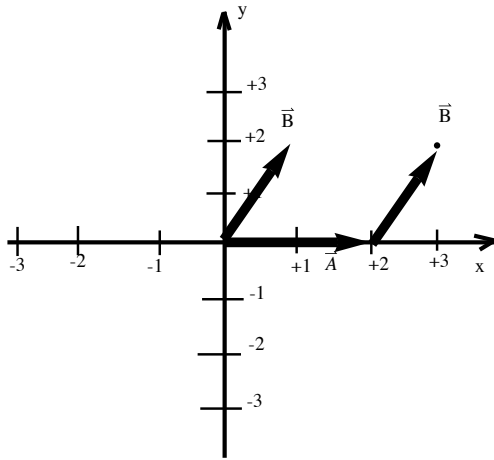
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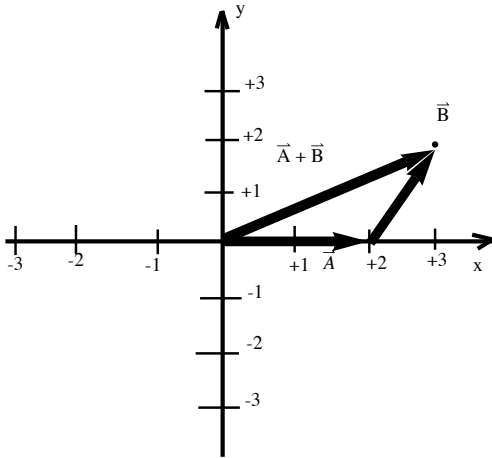## The Graphical Method of Adding Two Vectors Not at Right Angles:

(Also called the "Ruler/Protractor and Paper/Pencil Method".) Suppose there are two vectors $\vec{A}$

and $\vec{B}$ as indicated below and you wish to add them together to get $\vec{A} + \vec{B}$.

The first thing to do is move the vector $\vec{B}$ parallel to itself (so the angle does not change) until the

tail of $\vec{B}$ is a the tip of $\vec{A}$.

Remember this is the same vector $\vec{B}$ provided the length and angle remain the same. The

resultant vector $\vec{A} + \vec{B}$ is drawn from the tail of the first vector (that is $\vec{A}$) to the tip of the second

vector (that is $\vec{B}$).

The length of the vector $\vec{A} + \vec{B}$ can be measured with a ruler and the angle θ measured with a

protractor. The graphical method of adding vectors is useful to gain an intuition but mostly we will

use another method. This method uses the components of a vector.

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Rectangular Components of a Vector:

Consider the vector $\vec{C}$ indicated in the diagram below
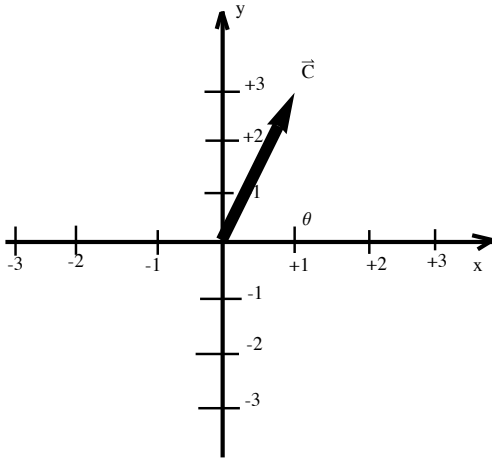
You can get vector $\vec{C}$ by adding on vector $\vec{C_x}$ in the x direction and having length 1 m to another vector $\vec{C_y}$ in the y direction and having a length 3 m.
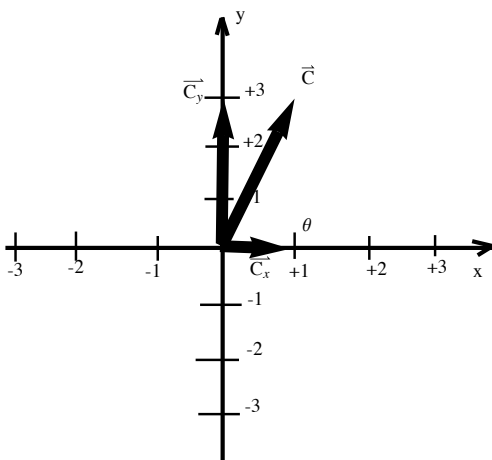
in other words, $\vec{C}=\vec{C_x}+\vec{C_y}$. $\vec{C_x}$ and $\vec{C_y}$ are called the rectangular components of the vector $\vec{C}$. In

the example above the length $|\vec{C_x}|=1$ and the length $|\vec{C_y}|=3$.

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Unit Vectors: $\hat{x}$ and $\hat{y}$

The unit vector $\hat{x}$ is a vector of length one meter in the x direction. The reason it is called a "unit"

vector is because the length of $\hat{x}$ is one. Similarly, the unit vector $\hat{y}$ is a vector of length one meter

in the y direction. The unit vectors $\hat{x}$ and $\hat{y}$ are drawn in the diagram below.
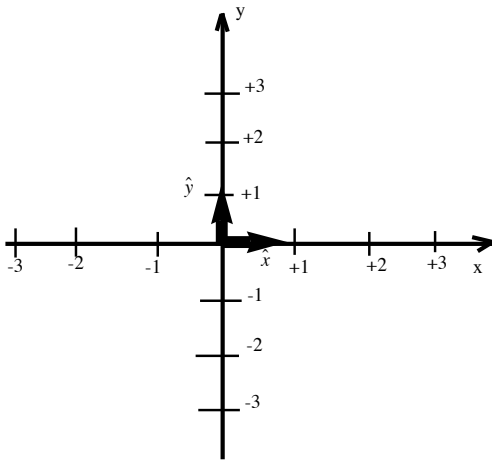
Using the unit vector $\hat{x}$ you can write $\vec{C_x} = C_x\,\hat{x}$ where C_x is the length $|\vec{C_x}|$ of the vector $\vec{C_x}$. In

the example above C_x=1 and $\vec{C_x} = 1\,\hat{x}$.

Similarly, using the unit vector $\hat{y}$ you can write $\vec{C_y}=C_y\,\hat{y}$ where C_y is the length $|\vec{C_y}|$ of the vector

$\vec{C_y}$. In the example above C_y=3 and $\vec{C_y}=3\,\hat{y}$.

Also you can write , $\vec{C}=C_x\,\hat{x}+C_y\,\hat{y}=1\,\hat{x}+3\,\hat{y}$ in this example.

NOTATION: Some books use $\hat{i}$ and $\hat{j}$ instead of $\hat{x}$ and $\hat{y}$ for the units vectors.

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Addition of Two Vectors Using Unit Vectors

Consider the addition of two vectors $\vec{A}$ and $\vec{B}$ not perpendicular to each other
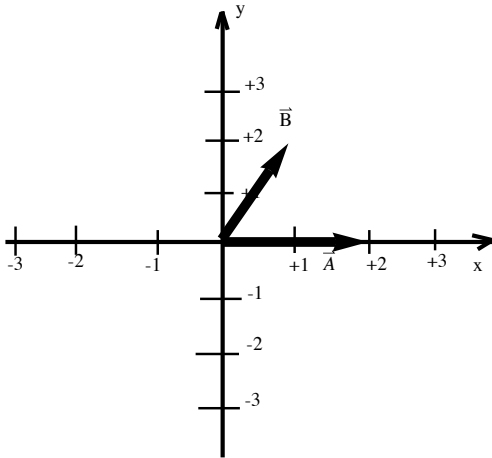
This time let us add the two vectors $\vec{A} + \vec{B}$ using the unit vector concept. First write vectors $\vec{A}$

and $\vec{B}$ using the unit vector notation and get $\vec{A}=2\,\hat{x}+0\,\hat{y}$ and $\vec{B}=1\,\hat{x}+2\,\hat{y}$. Next add the x-

components of vectors $\vec{A}$ and $\vec{B}$ and the y components of vectors $\vec{A}$ and $\vec{B}$ to obtain

$$\vec{A} + \vec{B} = \left(2\,\hat{x}+0\,\hat{y}\right) + \left(1\,\hat{x}+2\,\hat{y}\right) = \left(2+1\right)\hat{x} + \left(0+2\right)\hat{y} = 3\,\hat{x} + 2\,\hat{y}$$
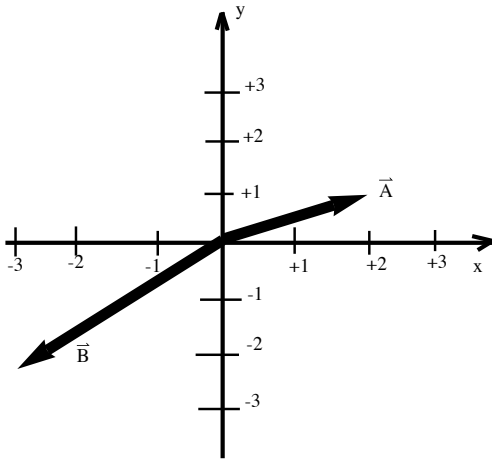
so the resultant vector is $\vec{A} + \vec{B} = 3\,\hat{x} + 2\,\hat{y}$. The length of the resultant vector (or sum) is

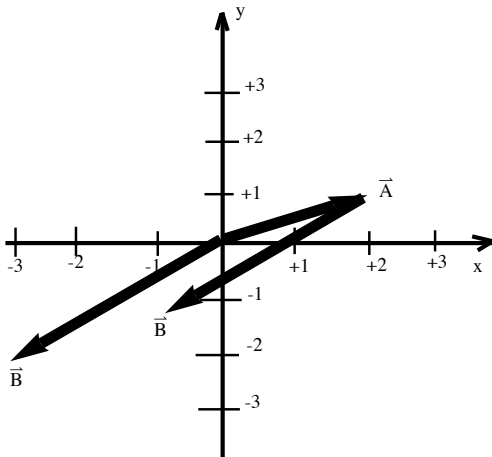$$\left|\vec{A}+\vec{B}\right| = \sqrt{3^2+2^2} = \sqrt{13} = 3.6$$

The angle θ that the vector makes with respect to the x axis is given by Tan$[\theta]=(2/3)=0.67$ so $\theta=34°$.

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Another Example: Adding Two Vectors $\vec{A}$ and $\vec{B}$

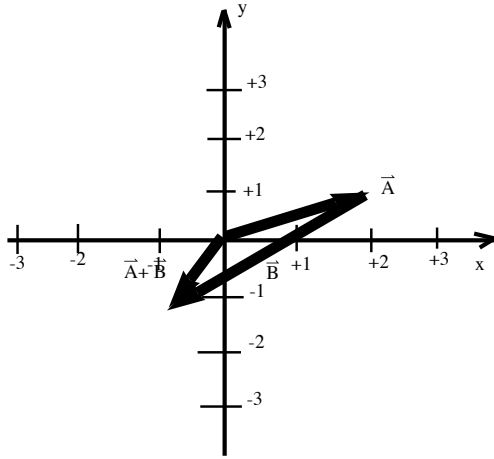First add the two vectors $\vec{A} + \vec{B}$ using the graphical (or ruler and protractor) method

Move the vector $\vec{B}$ parallel to itself until the tail of $\vec{B}$ is at the tip of $\vec{A}$ obtaining

The vector sum $\vec{A}+\vec{B}$ goes from the tail of $\vec{A}$ to the tip of $\vec{B}$

So from the diagram we see that the vector sum $\vec{A}+\vec{B}$ has an x-component of -1 and the y component of -1.

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You can get the same answer from the vector sum $\vec{B} + \vec{A}$ by first moving the vector $\vec{A}$ parallel to itself until the tail of $\vec{A}$ is at the tip of $\vec{B}$ obtaining
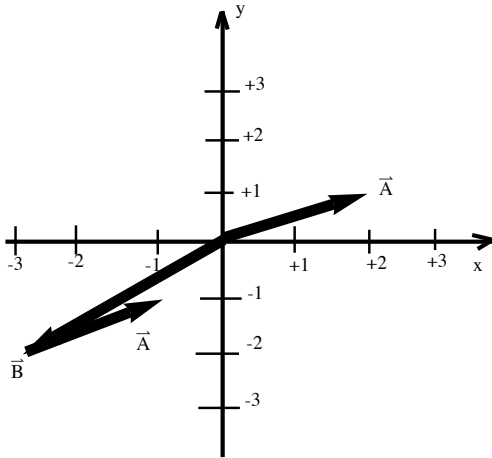
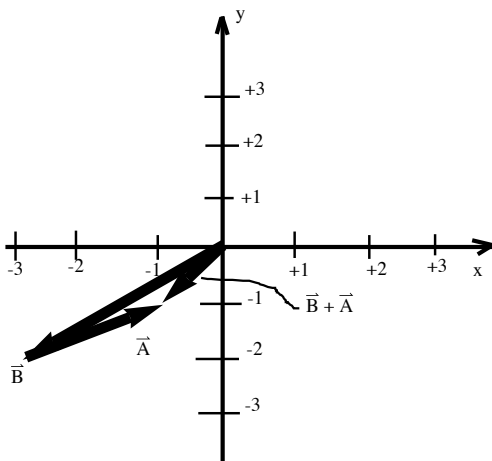The vector sum $\vec{B} + \vec{A}$ goes from the tail of $\vec{B}$ to the tip of $\vec{A}$

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The two vectors $\vec{A}$ and $\vec{B}$ above can also be added using components and unit vectors. First

write $\vec{A}$ and $\vec{B}$ in terms of their components obtaining get $\vec{A}=2\,\hat{x}+1\,\hat{y}$ and $\vec{B}=$ -3 $\hat{x}$ - 2 $\hat{y}$.

Next add the x-components of vectors $\vec{A}$ and $\vec{B}$ and the y components of vectors $\vec{A}$ and $\vec{B}$ to

obtain

$$\vec{A} + \vec{B} = \left(2\,\hat{x} + 1\,\hat{y}\right) + \left(-3\,\hat{x} - 2\,\hat{y}\right) = \left(2 - 3\right)\,\hat{x} + \left(1 - 2\right)\,\hat{y} = -1\,\hat{x} - 1\,\hat{y}$$

so the resultant vector is $\vec{A} + \vec{B} = -1\,\hat{x} - 1\,\hat{y}$. The length of the resultant vector (or sum) is

$$\left|\vec{A} + \vec{B}\right| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2} = 1.41$$

The angle θ that the vector makes with respect to the x axis is given by Tan[θ]=(-1)/(-1)=1 so θ=45°.

General Procedure for Adding Two Vectors using Components

Here a "cookbook" procedure is given for adding two vectors and obtaining the resultant vector,

$\vec{R} = \vec{A} + \vec{B}$.

1. Resolve the vector $\vec{A}$ into its components $\vec{A}_x$ and $\vec{A}_y$ so that $\vec{A} = \vec{A}_x + \vec{A}_y$.

2. Resolve the vector $\vec{B}$ into its components $\vec{B}_x$ and $\vec{B}_y$ so that $\vec{B} = \vec{B}_x + \vec{B}_y$.

3. Add the x-component of $\vec{A}_x$ to the x-components of $\vec{B}_x$ to obtain the x-component of the resultant $\vec{R}_x$ so $\vec{R}_x = \vec{A}_x + \vec{B}_x$.

4. Add the y-component of $\vec{A}_y$ to the y-components of $\vec{B}_y$ to obtain the y-component of the resultant $\vec{R}_y$ so $\vec{R}_y = \vec{A}_y + \vec{B}_y$.

5. Add the x-component of $\vec{R}_x$ to the y-components of $\vec{R}_y$ to obtain the resultant $\vec{R}$ so $\vec{R} = \vec{R}_x + \vec{R}_y$.

Technically you are now done but you might also be asked to

6. Component the magnitude of the resultant $|\vec{R}| = \sqrt{R_x{}^2 + R_y{}^2}$ and/or

7. The angle θ of the resultant using $\theta = \text{ArcTan}[y/x]$.

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Vectors in Other Dimensions

One Dimension: A vector in one dimension is just one number but usually a vector in one dimension is called a scalar. It has a magnitude (length) and direction (the sign) but usually this is consider just one number which can have positive and negative values.

 Three Dimensions*:* A vector $\vec{A}$ in three dimension has three components: A_x, A_y, and A_z . There are three unit vectors in three dimensions namely $\hat{x}$, $\hat{y}$, and $\hat{z}$ and $\vec{A}$ can be written as $\vec{A}$ $=A_x\hat{x}+A_y\hat{y}+A_z\hat{z}$ in three dimensions.   We will seldom discuss motion in three dimensions because two dimensions has most of the complications of three dimensional problem. Also, it is more difficult to draw three dimensional figures.

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Some Trigonometric Functions You Need to Know: Review

Trigonometric functions are tabulated relations that allow you to use the concept of similar triangles easily in physics. Similar triangles are two triangles of different size but have the same three angles. The right triangle below has a 90° angle by definition. An angle of 90° angle is the same as an angle of $\pi/2$ Radians.   The other two angles of the triangle add to $\alpha+\theta=90°$. The side h is opposite the 90° angle and is known as the hypotenuse.
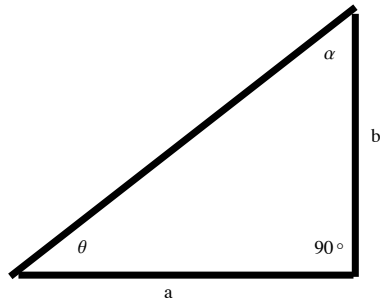
The **Tangent** function (or **Tan** for short) given by

$$\text{Tan}[\theta] = \frac{b}{a} = \frac{\text{opposite side}}{\text{adacent side}}$$

where b is the side **opposite** the θ angle and a is the side **adjacent** or next to the θ angle. The

Sine (or **Sin** for short) function is given by

$$\text{Sin}[\theta] = \frac{b}{h} = \frac{\text{opposite side}}{\text{hypotenuse}}$$

and the **Cosine** (or **Cos** for short) function is given by

$$\text{Cos}[\theta] = \frac{a}{h} = \frac{\text{adacent side}}{\text{hypotenuse}}$$

Also the **Pythagorean Theorem** is the relation

$$h^2 = a^2 + b^2$$

The inverse trignometric functions are call ArcTan, ArcSin, and ArcCos. They are equivalent to the InverseTangent, InverseSine, and InverseCos functions. On some calculators these inverse functions are written Tan^{-1}, Sin^{-1}, and Cos^{-1} respectively.

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