Lecture #5: Projectile Motion

Introduction and Summary:

1. Thus far in this course we have discussed the motion of objects only in one dimension.

Example: an object falling straight downward toward the Earth.

2. One dimensional motion does occur, often motions appearing in nature are in two and three dimensions.

3. An object moving in two dimensions in air is called a **projectile**.

Examples of projectile motion:

i) A baseball hit by a bat

ii) The motion of a ball after it has rolled off a table top.

GENERAL FEATURES OF PROJECTILE MOTION:

Vertical motion: Gravity produces accelerated motion of the projectile.

Horizontal motion: The horizontal motion is not accelerated.

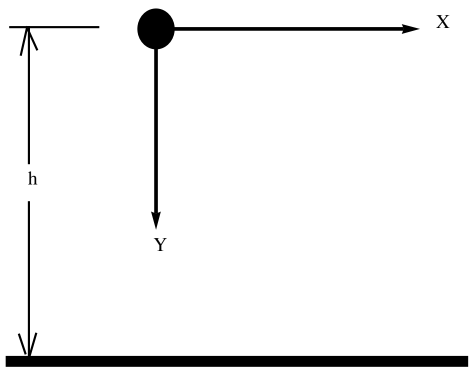
The vertical and horizontal motions of a projectile are INDEPENDENT of each other.

This is not a general feature of two and three dimensional motion.

Example: The Moon in orbit about the Earth has x and y motions interconnected.

Review: An Object Falling Straight Downward and Acted on by Gravity.

Consider the situation pictured below of a ball initially at rest v[0]=0 at the origin so y[0]=0 of coordinates and the positive y axis is downward:



Suppose we know the height h=15 m of the ball above the ground initially at t=0 and we want to calculate the time it takes the ball to hit the ground. Using $\Delta y = v_1 t + \frac{1}{2} g t^2$ with g=9.8 m/s^2, Δy=h=15 m, and v_1=0 yields

$$t = \sqrt{\frac{2\,h}{g}} = \sqrt{\frac{2 * 15\ \text{m}}{9.8\ \text{m}/\text{s}^2}} = 1.75\ \text{s}$$

$$\sqrt{\frac{2 * 15.}{9.8}}$$

1.74964

so the time it takes to fall and hit the ground is 1.75 s. Notice how the units cancel. The velocity when the ball hits the ground is obtained using $v_2{}^2 = v_1{}^2 + 2\,g\,\Delta y$ with v_1=0 and Δy=h. Thus solving for v_2 yields 17.1 m/s

$$v_2 = \sqrt{2\,g\,h} = \sqrt{2 * 9.8\ \text{m}/\text{s}^2 * 15\ \text{m}} = 17\ \text{m}/\text{s}$$

```
√(2 * 9.8 * 15.)
17.1464
```

How would things change if the positive y direction is upward?

◄ | ►

ANSWER:

1. $\Delta y = y_2 - y_1 = -15$ m would be negative since $y_1 = 0$ and $y_2 = -15$ m.

2. Also g = -9.8 m/s^2 would be negative in this case, since it is downward in the minus y direction.

The calculation of the time becomes

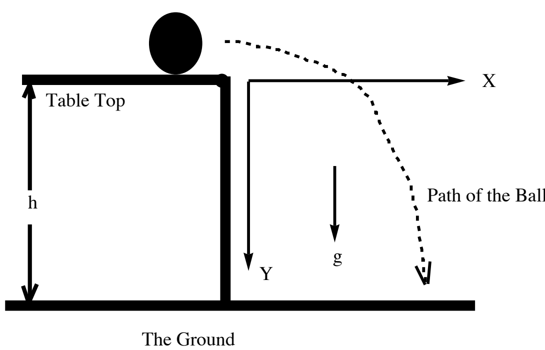
$$t = \sqrt{\frac{2\,h}{g}} = \sqrt{\frac{2 * (-15\,m)}{-9.8\,m/s^2}} = 1.75\,s$$

and again you get t=+1.75 s. You should work out what happens to the velocity calculation.

So how does moving in two dimensions change things?

Example: A Ball Rolling Off of a Table Top

A picture of the physical situation in this case appears below. Suppose the horizontal velocity of the ball is $v_x[0]$=20 m/s just as the ball leaves the table top and the vertical velocity is zero initially $v_x[0]$= 0 m/s. Again g=9.8 m/s^2 is positive since the vector acceleration of gravity is downward and because g then has a positive y component and also Δy=15 m is positive.



Notice the positive y direction is downward in the above diagram. We could just as easily solve this problem with the positive y direction upward and we will do so later.

Question:

How long does it take the ball to hit the ground?

It seems like it will take longer to hit the ground than in the case where the ball goes straight down.

Why? The ball has to travel a greater distance until it hits the ground.

A Better Answer: The ball takes the same time to hit the ground!

Perhaps surprisingly it takes the same amount of time as when the ball just fell vertically!

REASON: T**he x motion is independent of the y motion** for projectiles.

The Horizontal Motion: V_x=20 m/s is constant

$$X = V_x\, t \tag{1}$$

The Vertical Motion: The motion in the y direction is accelerated by gravity so

$$Y = V_y[0]\, t + \frac{1}{2} g\, t^2$$

Initially the ball is rolling only in the horizontal direction, thus $V_y[0]$=0 so

$$\triangle Y = \frac{1}{2} g \triangle t^2 \tag{2}$$

Given Y=h=15 m and g= 9.8 m/s^2 the time t to hit the ground it

$$t = \sqrt{\frac{2 \triangle Y}{g}} = 1.75\ s \tag{3}$$

$$\sqrt{\frac{2 * 15.}{9.8}}$$

1.74964

So the time t=1.75 s for the ball to hit the ground is the same both in this case where the ball moves horizontally and the case of the ball falling straight down.

Return to the Horizontal Motion: Use t=1.75 s to get the distance traveled in the x-direction

$$X = 20\, m/s * 1.75\, s = 35\, m \tag{4}$$

20. * 1.75

35.

The Shape of the Curve for the Projectile: Solving equation (1) for t we obtain

$$t = \frac{\triangle X}{V_x} \tag{5}$$

and next use equation (5) to eliminate t in equation (2)

$$Y = \frac{g}{2 V_x^2} X^2$$

This is the equation for a **parabola**. Using the numerical values g=9.8 and $V_x[0]$=20. equation (5) becomes

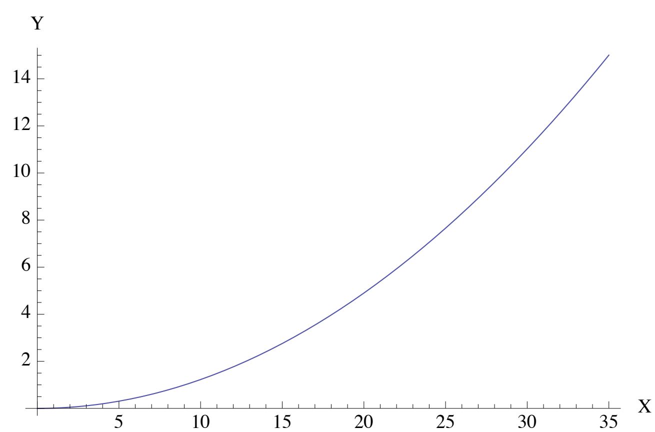
$$\frac{9.8\,m/s^2}{2*(20\,m/s)^2}$$

$$\frac{0.01225}{m}$$

$$Y = 0.012\ X^2 \qquad (6)$$

Graph of the Trajectory:

```
y[x_] = 0.01225 * x^2;
Plot[y[x], {x, 0, 35.}, AxesLabel → {"X", "Y"}]
```



So how does the ball hit the ground at the same time as a ball falling only in the vertical direction?

The basic reason the two balls hit the ground at the same time is that the ball is moving faster in the case of the parabolic motion when compared with the straight down motion.

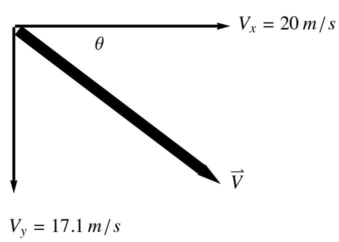
The Velocity of the Ball just before it hits the ground: The horizontal velocity V_x=20 m/s is always the in this problem. The vertical velocity starts out at zero $V_y[0]$=0 but increases to $V_y[t]$ which is calculated using $V_y[t]^2 = V_y[0]^2 + 2\,g\,Y = 2\,g\,Y$. Solving for $V_y[t]$ yields

$$V_y[t] = \sqrt{2\,g\,Y}$$

and using *Mathematica* we obtain

```
√(2 * 9.8 * 15)
17.1464
```

So the vertical velocity is 17.1 m/s and the horizontal velocity is 20 m/s just before the ball hits the ground. A picture of the velocity vector just before the ball hits the ground appears



The total velocity $\vec{V}$ is the vector sum of the x-component of velocity $\vec{V}_x$ and the y-component of velocity $\vec{V} = \vec{V}_x + \vec{V}_y$. The length or magnitude of the velocity vector is $|\vec{V}| = \sqrt{V_x^2 + V_y^2}$ and this is numerically $|\vec{V}|$ =26.3 m/s using *Mathematica*

```
√(20^2 + 17.1^2)
26.3137
```

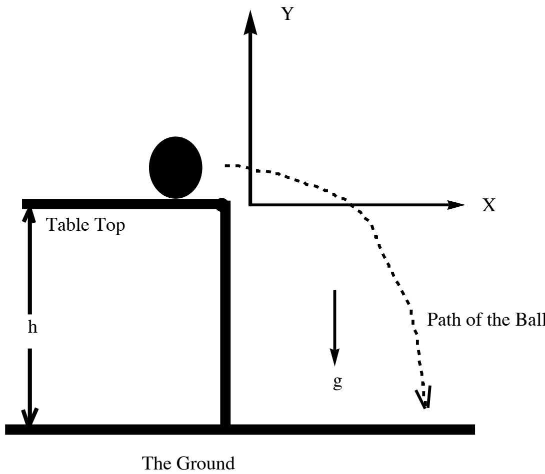
The angle θ downward with respect to the horizontal is computed using $\text{Tan}[\theta] = \frac{V_y}{V_x}$ or θ=ArcTan[$\frac{V_y}{V_x}$] = 40.5° using *Mathematica*

```
ArcTan[17.1/20] * (360/(2 π))
40.5304
```

The factor (360/2π) converts from the Radian angular measure to the degree measure since there are 360° in one revolution of a circle and this is equivalent to 2π Radians.

Conclusion: The magnitude of the velocity of the ball is 26.3 m/s for the parabolic motion just before it hits the ground and this greater than the 17.1 m/s for the velocity of the ball that falls straight downward.

Changing the Positive y Axis to **Upward**:



Notice the figure above has the positive y direction upward. What changes from the solution before?

◄ | ►

ANSWER:

The acceleration of gravity is negative now so g= - 9.8 m/s^2 also Y= -15 m is downward so it is negative. The vertical motion is described by $Y = \frac{1}{2} g t^2$ as before but now after substitution (-15 m)= $\frac{1}{2} \left(-9.8 \, m / s^2\right) t^2$ so the two minus signs cancel and solving we get $t = 1.75 \, s$ the same as before. The motion in the x-direction $X = V_x t$ is not effected by the y axis so everything is unchanged in the x direction.

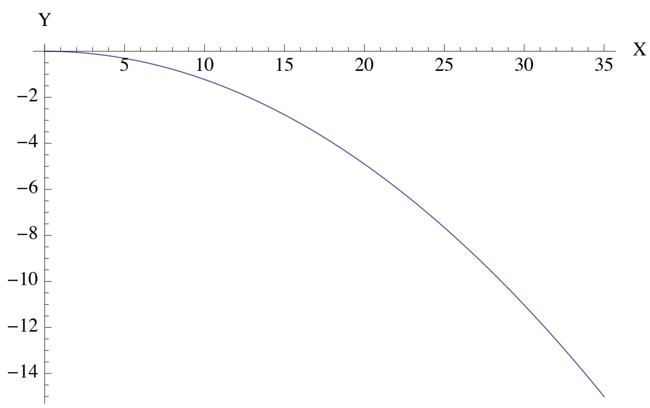
◄ | ►

The Shape of the Curve for the Projectile: Solving the x motion equation yields $t = \frac{X}{V_x}$ the same as before and using this in the y motion equation yields $Y = \frac{1}{2} g t^2 = \frac{g}{2 V_x^2} X^2$ in other words, $Y = \frac{g}{2 V_x^2} X^2$ which looks the same as plus y was down. However now g is negative g=-9.8 so the graph of the motion X versus Y appears

$$\frac{-9.8\,m/s^2}{2*(20\,m/s)^2}$$

$$-\frac{0.01225}{m}$$

```
y[x_] = -0.01225 * x^2;
gBigGraph = Plot[y[x], {x, 0, 35.}, AxesLabel → {"X", "Y"}]
```



which is what you should expect with the positive y direction upward since there are increasing negative values of y.

Suppose the value of g was decreased to half its value g= - 4.9 m/s^2 like on the Moon?

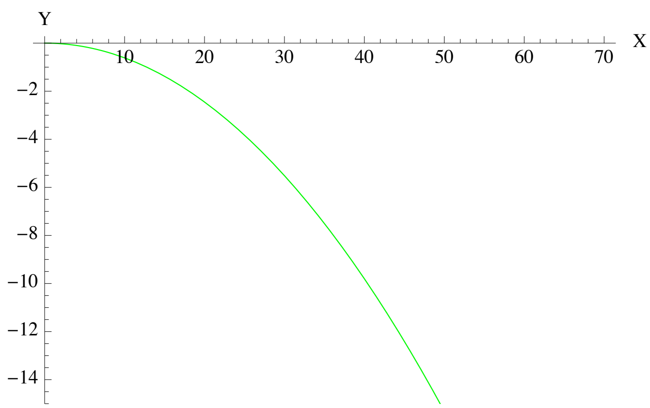
ANSWER:

As before, $Y = \frac{g}{2\,V_x^2}\,X^2$ but now the coefficient $\frac{g}{2\,V_x^2}$ has the value

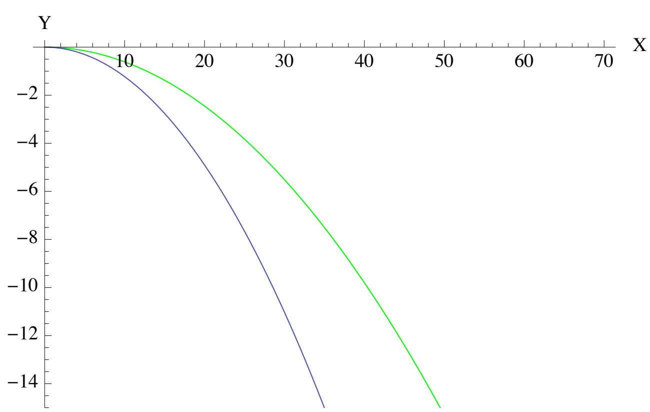
```
(-9.8/2 m/s^2) / (2 * (20 m/s)^2)
```

$-\frac{0.006125}{m}$

```
y[x_] = -0.006125 * x^2;
gLittleGraph =
 Plot[y[x], {x, 0, 70.}, PlotRange → {0, -15}, AxesLabel → {"X", "Y"}, PlotStyle → Green]
```




```
Show[gLittleGraph, gBigGraph]
```



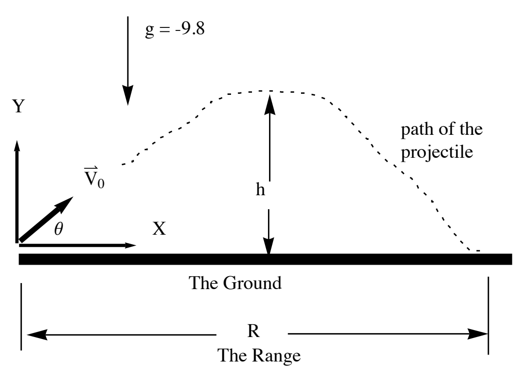
So if g is smaller, the ball travels further in the x direction before it hits the ground.

The Range R of a Projectile:

Another kind of projectile problem is picture below. The ground is flat throughout the trajectory of the projectile. Also suppose

1. the magnitude of the initial velocity is $V_0 = 35\ m/s$ and

2. the direction is initially an angle θ=30° above the horizontal direction. (Obviously, these numerical values can be different.)



What you would like to calculate:

1. the range R of the projectile which is the maximum distance traveled in the horizontal direction or x direction and

2. the maximum height in the y direction is labeled h.

The Key Idea: The motion in the x-direction is independent of the motion in the y-direction for projectile motion.

Consequences:

1. The motion in the x-direction has constant velocity since gravity does not act horizontally. (Air resistance is neglected for simplicity.)

2. The motion in the y-direction always has an acceleration g = -9.8 m/s^2 going up, coming down, and at the top of the trajectory. The minus sign is because the vector $\vec{g} = -9.8\ \hat{x}$ is pointed downward toward the center of the Earth (which produces the gravitational acceleration) and this is opposite the direction of the positive y-axis.

3. The initial velocity of the projectile has an x-component of $V_{x0} = V_0 \text{Cos}[\theta]$ and a y-component of $V_{y0} = V_0 \text{Sin}[\theta]$.

```
Vx0 = 35 m / s * Cos[30. °]
```

$$\frac{30.3109\ m}{s}$$

```
Vy0 = 35 m / s * Sin[30. °]
```

$$\frac{17.5\ m}{s}$$

◄ | ►

Calculating the Trajectory of the Projectile:

The x-motion: $x = V_{x0}\ t$

The y-motion: $y = V_{y0}\ t + \frac{1}{2}\left(-9.8\ m/s^2\right) t^2$

with $\mathbf{V_{x0}}$**=30.3 m/s** and $\mathbf{V_{x0}}$**=17.5 m/s**.

Solve the x motion equation for time $t = \frac{x}{V_{x0}}$ and then use this to eliminate t in the y motion equation

obtaining

$$y = \frac{V_{y0}}{V_{x0}}\ x + \frac{1}{2}\ \frac{g}{V_{x0}{}^2}\ x^2$$

The coefficients have numerical values

$$\frac{V_{y0}}{V_{x0}}$$

```
0.57735
```

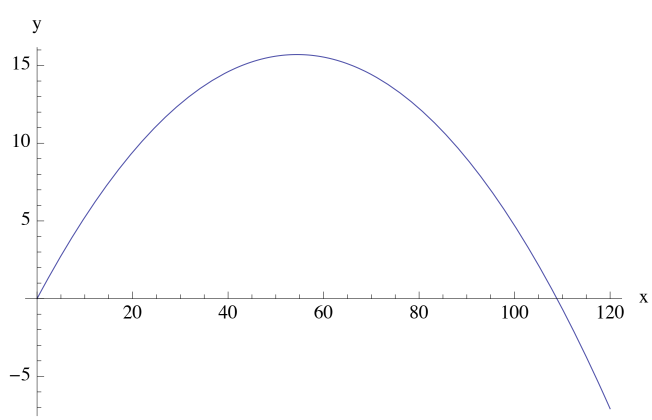
$$g = -9.8\ m/s^2;$$

$$\frac{g}{2 * V_{x0}{}^2}$$

$$-\frac{0.00533333}{m}$$

so a graph of the trajectory is parabolic

```
y[x_] := 0.577 * x - 0.0053 * x^2;
Plot[y[x], {x, 0, 120}, AxesLabel → {"x", "y"}]
```



The maximum value of x in the plot function is 110 m and this was obtained by trial and error. The actual range R is a little less than 110 m according the to the graph above but later we will get it more precisely.

What is the maximum height h according to the graph above?

Calculation of Maximum Height h in the y-direction: Use the kinematic equation

$$V_2{}^2 = V_1{}^2 + 2\,a\,\Delta y \qquad (7)$$

with Δy=h, $v_1 = V_{y0}$, $a = -g = -9.8\ m/s^2$, $v_2 = 0\ m/s$ because the y-component of velocity is zero at the top of the trajectory. Thus equation (7) becomes

$$0 = V_{y0}{}^2 + 2\,g\,h \qquad (8)$$

and solving for h yields

$$h = \frac{Vy_0{}^2}{2\,g} \qquad (9)$$

$$\mathbf{h = -\frac{V_{y0}{}^2}{2 * g}}$$

15.625 m

This value of h agrees (more or less) with the graph above.

Calculation of y-component of velocity when the Projectile Hits the Earth: On the way down, h is the same as on the way up. Also a = -g just as on the way up. The initial y-component of velocity v_1=0 at the top of the trajectory thus equation (7) becomes on the way down

$$v_2{}^2 = 0 + 2\,g\,h \tag{10}$$

and solving for the y-component of velocity just as the projectile hits the Earth from equation (10)

$$v_2 = \sqrt{2\,g\,h} \tag{11}$$

Since the h is the same size as on the way up, v_2 is the same magnitude as v_1 on the way up but the direction is downward here and this is the reverse of the up case. The sign of both g and h is negative in

$$\sqrt{2 * g * h}$$

$$17.5 \sqrt{\frac{m^2}{s^2}}$$

which agrees with what we got above.

Calculation of the Time up: The time up is determined by ΔV=aΔt where

(v_2-v_1)=-V_{y0} is negative as is g = -9.8 m/s^2 so the computed time Δt is positive as it must be. Thus

$$\Delta t = \frac{0 - V_{y0}}{g}$$

1.78571 s

Calculation of the Time down:

The velocity at the top v_1=0 on the way down and we just got v_2=-17.5 m/s (the minus is because it is downward). Also a = -9.8 m/s^2 so the time from top to bottom is

$$\Delta t = \frac{-V_{y0}}{g}$$

1.78571 s

which is the same as the time up. The total time in the air is

```
tTotal = Δt + Δt
```

```
3.57143 s
```

Calculation of the Range R:

```
R = Vx0 * tTotal
```

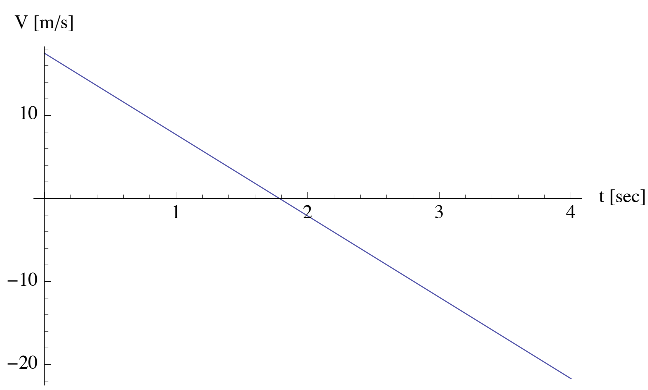
```
108.253 m
```

This agrees with the picture of the trajectory above.

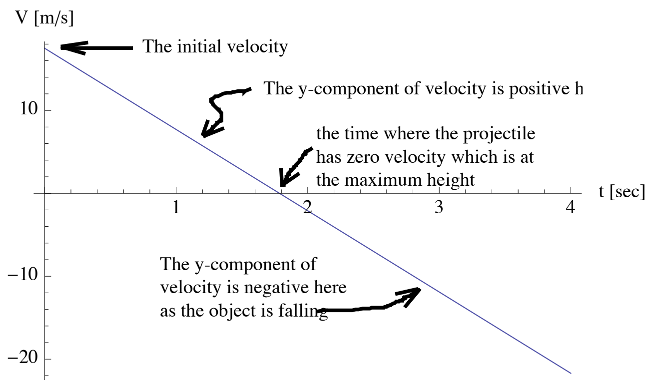
The Velocity versus Time for the Entire Trajectory:

Recall the kinematic equation $v_2 = v_1 + g\ t$. Use the initial y-component of velocity V_{y0} for v_1 and regard v_2 as a function of time t. Note the acceleration g = - 9.8 m/s^2.

```
Clear[t, Vy0, g]; Vy0 = 17.5; g = -9.8;
V[t_] = Vy0 + g * t;
Plot[V[t], {t, 0, 4}, AxesLabel → {"t [sec]", "V [m/s]"}]
```



Notice the slope of the graph of V[t] versus time t is negative g = -9.8 The point t=0 is the initial velocity 17.5 and positive since the projectile is moving upward away from the Earth. Notice as well that about t=1.8 s the velocity V is zero even though the acceleration is NOT zero. For times greater than 1.8 s the velocity increases in a negative sense since the projectile is moving downward toward the Earth.



The graph should stop when the object hits the ground. What is that time?