# Distinguishing mixed quantum states: Minimum-error discrimination versus optimum unambiguous discrimination

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We consider two different optimized measurement strategies for the discrimination of nonorthogonal quantum states. The first is ambiguous discrimination with a minimum probability of inferring an erroneous result, and the second is unambiguous, i.e., error-free, discrimination with a minimum probability of getting an inconclusive outcome, where the measurement fails to give a definite answer. For distinguishing between two mixed quantum states, we investigate the relation between the minimum-error probability achievable in ambiguous discrimination, and the minimum failure probability that can be reached in unambiguous discrimination of the same two states. The latter turns out to be at least twice as large as the former for any two given states. As an example, we treat the case where the state of the quantum system is known to be, with arbitrary prior probability, either a given pure state, or a uniform statistical mixture of any number of mutually orthogonal states. For this case we derive an analytical result for the minimum probability of error and perform a quantitative comparison with the minimum failure probability.

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## I. INTRODUCTION

Stimulated by the rapid developments in quantum communication and quantum cryptography, the question as to how to optimally discriminate between different quantum states has gained renewed interest [1]. The problem is to determine the actual state of a quantum system that is prepared, with given prior probability, in a certain but unknown state belonging to a finite set of given possible states. When the possible states are not mutually orthogonal, it is impossible to devise a measurement that can distinguish between them perfectly. Therefore optimum measurement strategies have been developed with respect to various criteria.

Recently much work has been devoted to the strategy of optimum unambiguous discrimination. Here it is required that, whenever a definite outcome is returned after the state-distinguishing measurement, the result should be error free, i.e., unambiguous. This can be achieved at the expense of allowing for a nonzero probability of inconclusive outcomes, where the measurement fails to give a definite answer. When the probability of failure is minimum, optimum unambiguous discrimination is realized. Analytical solutions for the minimum failure probability $Q_F$ have been found for distinguishing between two [2–5] and among three [6–8] arbitrary pure states, and between any number of pure states that are symmetric and equiprobable [9]. On the other hand, the investigation of unambiguous discrimination involving mixed states, or sets of pure states, respectively, started only recently [10–16]. So far exact analytical results are known only for simple cases [11–14]. In addition, for unambiguously discriminating between two arbitrary mixed states, general upper and lower bounds have been derived for the minimum failure probability [14].

In contrast to unambiguous discrimination, the earliest measurement strategy for distinguishing nonorthogonal quantum states requires that a definite, i.e., conclusive outcome is to be returned in each single measurement. This means that errors in the conclusive result are unavoidable and the discrimination is ambiguous. Based on the outcome of the measurement, a guess is made as to what the state of the quantum system was. The optimum measurement then minimizes the probability of errors, i.e., the probability of making a wrong guess. For distinguishing two mixed quantum states, a general expression for the minimum achievable error probability $P_E$ has been derived in the pioneering work by Helstrom [17]. When more than two given states are involved, an analytical solution is known only for a restricted number of cases, the most important of them being the case of equiprobable and symmetric states that are either pure [18] or mixed [19,20]. Finally it is worth mentioning that the original minimum-error discrimination strategy has been extended to determine the minimum achievable probability of errors under the condition that a fixed finite probability of inconclusive outcomes is allowed to occur [21,22], giving no definite result.

In the present contribution we investigate the relation between the minimum-error probability $P_E$ for ambiguously distinguishing two mixed quantum states, and the minimum failure probability $Q_F$ attainable in unambiguous discrimination of the same two states. In Sec. II we show that for two arbitrary mixed quantum states the latter is always at least twice as large as the former. As an analytically solvable special example, in Sec. III we treat the problem of deciding whether the state of the quantum system is either a given pure state, or a mixed state being a uniform statistical mixture of any number of mutually orthogonal states. First we derive an analytical expression for the minimum-error probability in this example, extending a previous result [23] to

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the case of arbitrary prior probabilities. We then perform a comparison with unambiguous discrimination by making use of the general solution for the minimum failure probability in unambiguous quantum state filtering out of an arbitrary number of states [12]. Note that in our preceding work we considered state discrimination involving mixed states in the context of distinguishing between two sets of pure states, referring to the discrimination problem as filtering [11,12,24] when the first set contains only a single state. Apart from being an illustration for the general relation between $P_E$ and $Q_F$, our specific example is of interest on its own for applications that are mentioned in the conclusions.

## II. INEQUALITY FOR THE MINIMUM PROBABILITIES OF ERROR AND OF FAILURE

In the frame of the quantum detection and estimation theory [17], a measurement that discriminates between two mixed states, described by the density operators $\rho_1$ and $\rho_2$, and occurring with the prior probabilities $\eta_1$ and $\eta_2 = 1-\eta_1$, respectively, can be formally described with the help of two detection operators $\Pi_1$ and $\Pi_2$. These operators are defined in such a way that $\mathrm{Tr}(\rho\Pi_j)$ is the probability to infer the system is in the state $\rho_j$ if it has been prepared in the state $\rho$. Since the probability is a real non-negative number, the detection operators have to be Hermitean and positive semidefinite. In the error-minimizing measurement scheme the measurement is required to be exhaustive and conclusive in the sense that in each single case with certainty one of the two possible states is identified, although perhaps incorrectly, while inconclusive results allowing no identification do not occur. This leads to the requirement

$$\Pi_1 + \Pi_2 = I_{D_S}, \tag{1}$$

where $I_{D_S}$ denotes the unit operator in the $D_S$ dimensional physical state space of the quantum system under consideration. The overall probability $P_{\mathrm{err}}$ to make an erroneous guess for any of the incoming states is then given by

$$P_{\mathrm{err}} = 1 - \sum_{j=1}^{2} \eta_j \mathrm{Tr}(\rho_j \Pi_j) = \eta_1 \mathrm{Tr}(\rho_1 \Pi_2) + \eta_2 \mathrm{Tr}(\rho_2 \Pi_1), \tag{2}$$

where use has been made of the relation $\eta_1 + \eta_2 = 1$. In order to find the strategy for minimum-error discrimination, one has to determine the specific set of detection operators that minimizes the value of $P_{\mathrm{err}}$ under the constraint given by Eq. (1). As found by Helstrom [17], the smallest achievable error probability $P_{\mathrm{err}}^{\min} = P_E$ is given by

$$P_E = \tfrac{1}{2}(1 - \mathrm{Tr}|\eta_2\rho_2 - \eta_1\rho_1|), \tag{3}$$

where $|\sigma| = \sqrt{\sigma^\dagger \sigma}$ for any operator $\sigma$.

While the original derivation of Eq. (3) relies on variational techniques, for the purpose of this paper it is advantageous to analyze the two-state minimum-error measurement with the help of an alternative method [25,26]. To this end we express Eq. (2) alternatively as

$$P_{\mathrm{err}} = \eta_1 + \mathrm{Tr}(\Lambda \Pi_1) = \eta_2 - \mathrm{Tr}(\Lambda \Pi_2), \tag{4}$$

where we introduced the Hermitean operator

$$\Lambda = \eta_2\rho_2 - \eta_1\rho_1 = \sum_{k=1}^{D_S} \lambda_k |\phi_k\rangle\langle\phi_k|. \tag{5}$$

Here the states $|\phi_k\rangle$ denote the orthonormal eigenstates belonging to the eigenvalues $\lambda_k$ of the operator $\Lambda$. By using the spectral decomposition of $\Lambda$, we get the representations [23]

$$P_{\mathrm{err}} = \eta_1 + \sum_{k=1}^{D_S} \lambda_k \langle\phi_k|\Pi_1|\phi_k\rangle = \eta_2 - \sum_{k=1}^{D_S} \lambda_k \langle\phi_k|\Pi_2|\phi_k\rangle. \tag{6}$$

The eigenvalues $\lambda_k$ are real, and without loss of generality we can number them in such a way that

$$\lambda_k < 0 \quad \text{for} \quad 1 \leqslant k < k_0,$$

$$\lambda_k > 0 \quad \text{for} \quad k_0 \leqslant k \leqslant D,$$

$$\lambda_k = 0 \quad \text{for} \quad D < k \leqslant D_S. \tag{7}$$

The optimization task is then to determine the specific operators $\Pi_1$, or $\Pi_2$, respectively, that minimize the right-hand side of Eq. (6) under the constraint that

$$0 \leqslant \langle\phi_k|\Pi_j|\phi_k\rangle \leqslant 1 \quad (j=1,2) \tag{8}$$

for all eigenstates $|\phi_k\rangle$. The latter requirement is due to the fact that $\mathrm{Tr}(\rho\Pi_j)$ denotes a probability for any $\rho$. From this constraint and from Eq. (6) it immediately follows that the smallest possible error probability, $P_{\mathrm{err}}^{\min} \equiv P_E$, is achieved when the detection operators are chosen in such a way that the equations $\langle\phi_k|\Pi_1|\phi_k\rangle = 1$ and $\langle\phi_k|\Pi_2|\phi_k\rangle = 0$ are fulfilled for eigenstates belonging to negative eigenvalues, while eigenstates corresponding to positive eigenvalues obey the equations $\langle\phi_k|\Pi_1|\phi_k\rangle = 0$ and $\langle\phi_k|\Pi_2|\phi_k\rangle = 1$. Hence the optimum detection operators are given by

$$\Pi_1 = \sum_{k=1}^{k_0-1} |\phi_k\rangle\langle\phi_k|, \qquad \Pi_2 = \sum_{k=k_0}^{D} |\phi_k\rangle\langle\phi_k|, \tag{9}$$

where these expressions have to be supplemented by projection operators onto eigenstates belonging to the eigenvalue $\lambda_k = 0$, in such a way that $\Pi_1 + \Pi_2 = I_{D_S}$. Using Eq. (2), from the optimum detection operators the minimum-error probability is found to be [23]

$$P_E = \eta_1 - \sum_{k=1}^{k_0-1} |\lambda_k| = \eta_2 - \sum_{k=k_0}^{D} |\lambda_k|. \tag{10}$$

By taking the sum of these two alternative representations, using $\eta_1 + \eta_2 = 1$, we arrive at

$$P_E = \frac{1}{2}\left(1 - \sum_k |\lambda_k|\right) = \frac{1}{2}(1 - \mathrm{Tr}|\Lambda|), \tag{11}$$

which is equivalent to Eq. (3). Interestingly, for characterizing the measurement described by the detection operators given in Eq. (9), two different cases have to be considered. Provided that there are positive as well as negative eigenvalues in the spectral decomposition of $\Lambda$, the measurement obviously is a von Neumann measurement that consists of performing projections onto the two orthogonal subspaces spanned by the two sets of states $\{|\phi_1\rangle, \ldots, |\phi_{k_0-1}\rangle\}$ and $\{|\phi_{k_0}\rangle, \ldots, |\phi_D\rangle\}$. On the other hand, when negative eigenvalues do not exist it follows that $\Pi_1=0$ and $\Pi_2=I_{D_S}$, which means that the minimum-error probability can be achieved by always guessing that the quantum system is in the state $\rho_2$, without performing any measurement at all. Similar considerations hold true in the absence of positive eigenvalues. These findings are in agreement with the recent observation [27] that a measurement does not always aid minimum-error discrimination. In Sec. III we shall discuss a corresponding example.

In the error-minimizing scheme for discriminating two mixed states $\rho_1$ and $\rho_2$ of a quantum system, a nonzero probability of making a correct guess can always be achieved. However, it is obvious that the states can only be distinguished unambiguously when at least one of the mixed states contains at least one component, in the $D_S$ dimensional physical state space of the quantum system, that does not also occur in the other mixed state. As has been shown recently [14], the minimum failure probability in unambiguous discrimination, $Q_F$, obeys the inequality

$$Q_F \geqslant \begin{cases} 2\sqrt{\eta_1\eta_2}\, F(\rho_1,\rho_2) & \text{if} \quad \sqrt{\dfrac{\eta_{\min}}{\eta_{\max}}} \geqslant F, \\ \eta_{\min} + \eta_{\max}[F(\rho_1,\rho_2)]^2 & \text{otherwise}. \end{cases} \tag{12}$$

Here $\eta_{\min}$ ($\eta_{\max}$) is the smaller (larger) of the two prior probabilities $\eta_1$ and $\eta_2$, and $F$ is the fidelity, defined as

$$F(\rho_1,\rho_2) = \mathrm{Tr}[(\sqrt{\rho_2}\,\rho_1\sqrt{\rho_2})^{1/2}]. \tag{13}$$

Since the two lines of Eq. (12) are the geometric and the arithmetic mean, respectively, of the same expressions, it is clear that the first line denotes the overall lower bound $Q_L$ on the failure probability, i.e.,

$$Q_F \geqslant Q_L \equiv 2\sqrt{\eta_1\eta_2}\, F(\rho_1,\rho_2) \tag{14}$$

for arbitrary values of the prior probabilities.

In the following we want to compare the minimum-error probability $P_E$ given by Eq. (3) with the smallest possible failure probability that is achievable in a measurement designed for discriminating the two mixed states unambiguously. Our procedure will be closely related to the derivation of inequalities between the fidelity and the trace distance [28]. In order to estimate $Q_L$, or the fidelity, respectively, it is advantageous to use a particular orthonormal basis. It has been proven [25,28] that when the basis states are chosen to be the eigenstates $\{|l\rangle\}$ of the Hermitean operator $\rho_2^{-1/2}(\sqrt{\rho_2}\,\rho_1\sqrt{\rho_2})^{1/2}\rho_2^{-1/2}$, the fidelity takes the form

$$F(\rho_1,\rho_2) = \sum_l \sqrt{\langle l|\rho_1|l\rangle\langle l|\rho_2|l\rangle} = \sum_l \sqrt{r_l s_l}. \tag{15}$$

Here $\Sigma_l |l\rangle\langle l| = I$, with $I$ being the unit operator, and we introduced the abbreviations $r_l=\langle l|\rho_1|l\rangle$ and $s_l=\langle l|\rho_2|l\rangle$. The lower bound on the failure probability then obeys the equation

$$1 - Q_L = 1 - 2\sqrt{\eta_1\eta_2}\sum_l \sqrt{r_l\, s_l} = \sum_l (\sqrt{\eta_1 r_l} - \sqrt{\eta_2 s_l})^2, \tag{16}$$

where the second equality sign is due to the relation $\eta_1+\eta_2=1$ and to the normalization conditions $\mathrm{Tr}\rho_1=\Sigma_l\, r_l=1$ and $\mathrm{Tr}\rho_2=\Sigma_l\, s_l=1$.

Let us now estimate the minimum-error probability $P_E$, using the same set of basis states $\{|l\rangle\}$. Because of Eq. (11) and of the fact that $\langle\phi_k|\phi_k\rangle=\Sigma_l\, |\langle\phi_k|l\rangle|^2=1$, we can write

$$\begin{aligned} 1 - 2P_E &= \sum_k |\lambda_k| = \sum_l \sum_k |\lambda_k||\langle\phi_k|l\rangle|^2 \geqslant \sum_l \left|\sum_k \lambda_k |\langle\phi_k|l\rangle|^2\right| \\ &= \sum_l |\langle l|\Lambda|l\rangle|, \end{aligned} \tag{17}$$

where the last equality sign follows from the spectral decomposition of the operator $\Lambda$ [see Eq. (5)]. After reexpressing $\Lambda$ in terms of the density operators describing the given states, we arrive at

$$\begin{aligned} 1 - 2P_E &\geqslant \sum_l |\langle l|\,\eta_1\rho_1 - \eta_2\rho_2|l\rangle| \\ &= \sum_l |\sqrt{\eta_1 r_l} - \sqrt{\eta_2 s_l}||\sqrt{\eta_1 r_l} + \sqrt{\eta_2 s_l}|. \end{aligned} \tag{18}$$

By comparing the expressions on the right-hand sides of Eqs. (16) and (18) it becomes immediately obvious that $1-2P_E \geqslant 1-Q_L$, or $P_E \leqslant Q_L/2$, respectively. Together with Eq. (14) this implies our final result

$$P_E \leqslant \tfrac{1}{2} Q_F. \tag{19}$$

Hence for two arbitrary mixed states, occurring with arbitrary prior probabilities, the smallest possible failure probability in unambiguous discrimination is at least twice as large as the minimum probability of errors achievable for ambiguously distinguishing the same states.

## III. DISTINGUISHING BETWEEN A PURE STATE AND A UNIFORMLY MIXED STATE

For a quantitative comparison between the minimum probabilities of error and of failure we wish to consider a state discrimination problem that involves mixed states and that can be solved analytically with respect to the two different strategies under investigation. Minimum-error discrimination between two mixed states, or between two sets of states both consisting of a certain number of given pure states, respectively, has been recently treated analytically under the restriction that the total Hilbert space collectively spanned by the states is only two dimensional [24]. When the dimensionality $D$ of the relevant Hilbert space is larger than two, however, the explicit analytical evaluation of $P_E$ poses severe difficulties, due to the fact that applying the Helstrom formula amounts to calculating the eigenvalues of a $D$ dimensional matrix. In the following we consider a simple yet nontrivial discrimination problem where we are able to find an analytical solution for minimum-error discrimination in a Hilbert space of arbitrary many dimensions.

We assume that the quantum system is either prepared, with the prior probability $\eta_1$, in the pure state

$$\rho_1 = |\psi\rangle\langle\psi| \tag{20}$$

or, with the prior probability $\eta_2 = 1-\eta_1$, in a uniform statistical mixture of $d$ mutually orthonormal states, described by the density operator

$$\rho_2 = \frac{1}{d}\sum_{j=1}^{d} |u_j\rangle\langle u_j| = \frac{1}{d}I_d \tag{21}$$

with $\langle u_i|u_j\rangle = \delta_{ij}$ and $I_d$ denoting the unit operator in the $d$ dimensional Hilbert space $\mathcal{H}_d$ spanned by the states $|u_1\rangle, \ldots, |u_d\rangle$. It is convenient to introduce additional mutually orthogonal and normalized states $|v_0\rangle$ and $|v_1\rangle$ in such a way that

$$|\psi\rangle = \sqrt{1-\|\psi^{\parallel}\|^2}|v_0\rangle + \|\psi^{\parallel}\||v_1\rangle, \tag{22}$$

where $\|\psi^{\parallel}\|\,|v_1\rangle \equiv |\psi^{\parallel}\rangle$ is the component of $|\psi\rangle$ that lies in $\mathcal{H}_d$, i.e.,

$$\|\psi^{\parallel}\|^2 = \langle\psi^{\parallel}|\psi^{\parallel}\rangle = \sum_{j=1}^{d} |\langle u_j|\psi\rangle|^2. \tag{23}$$

The total Hilbert space spanned by the set of states $\{|\psi\rangle, |u_1\rangle, \ldots, |u_d\rangle\}$ is $d$ dimensional if $\|\psi^{\parallel}\| = 1$, and $(d+1)$ dimensional otherwise. With $D_S$ denoting the dimensionality of the physical state space of the quantum system under consideration, it is clear that the relations $D_S \geqslant d$ or $D_S \geqslant d+1$ have to be fulfilled in the former and the latter case, respectively.

In order to calculate the minimum-error probability $P_E$ with the help of Eq. (11), we have to determine the eigenvalues $\lambda$ of the operator $\Lambda = \eta_2\rho_2 - \eta_1\rho_1$. This amounts to solving the characteristic equation det $A=0$ with

$$A(\lambda) = \lambda I_{d+1} - \Lambda = \lambda I_{d+1} + \eta_1|\psi\rangle\langle\psi| - \frac{\eta_2}{d}I_d, \tag{24}$$

where the unit operator in $\mathcal{H}_{d+1}$ can be written as $I_{d+1} = |v_0\rangle\langle v_0| + I_d$. We now take advantage of the fact that by changing the basis system the unit operator in $\mathcal{H}_d$ can be alternatively expressed as $I_d = |v_1\rangle\langle v_1| + \Sigma_{j=2}^{d}|v_j\rangle\langle v_j|$ with $|v_1\rangle$ being given by Eq. (22) and $\langle v_i|v_j\rangle = \delta_{ij}$ for $i,j = 0,1,\ldots,d$. Therefore

$$A = \lambda|v_0\rangle\langle v_0| + \eta_1|\psi\rangle\langle\psi| + \left(\lambda - \frac{\eta_2}{d}\right)\sum_{j=1}^{d}|v_j\rangle\langle v_j|, \tag{25}$$

and by using the decomposition of $|\psi\rangle$ in this basis, Eq. (22), we readily obtain the matrix elements $A_{ij} = \langle v_i|A(\lambda)|v_j\rangle$. From the condition det $A=0$ the eigenvalues are found to be

$$\lambda_{1,2} = \frac{1}{2}\left[\frac{\eta_2}{d} - \eta_1 \mp \sqrt{\left(\frac{\eta_2}{d} + \eta_1\right)^2 - 4\eta_1\frac{\eta_2}{d}\|\psi^{\parallel}\|^2}\right],$$

$$\lambda_k = \frac{\eta_2}{d} \quad (k = 3, \ldots d+1). \tag{26}$$

Clearly, when the set of states $\{\langle\psi|, |u_1\rangle, \ldots, |u_d\rangle\}$ is linearly independent, i.e., for $\|\psi^{\parallel}\| < 1$, the square root in the first of the equations (26) is larger than $|\eta_2/d - \eta_1|$. Therefore $\lambda_1$ is the only negative eigenvalue and, according to Eq. (9), the detection operator $\Pi_1$ that determines the minimum-error measurement scheme is the projector onto the eigenstate belonging to the negative eigenvalue. Provided that $\eta_1 > \eta_2/d$, the same holds true for $\|\psi^{\parallel}\| = 1$. However, when $\|\psi^{\parallel}\| = 1$ and $\eta_1 \leqslant \eta_2/d$, a negative eigenvalue does not exist, implying that $\Pi_1 = 0$. This means that in the latter case one cannot find a measurement strategy that yields a smaller probability of errors than always guessing that the quantum system is prepared in the state $\rho_2$. By inserting the eigenvalues into Eq. (11) we finally arrive at the minimum-error probability

$$P_E = \frac{\eta_1}{2} + \frac{\eta_2}{2d} - \frac{1}{2}\sqrt{\left(\eta_1 + \frac{\eta_2}{d}\right)^2 - 4\eta_1\frac{\eta_2}{d}\|\psi^{\parallel}\|^2}. \tag{27}$$

For $d=1$, Eq. (27) reduces to the well-known Helstrom bound [17] for minimum-error discrimination between the two pure states $|\psi\rangle$ and $|u_1\rangle$.

Now we turn to unambiguous discrimination. The task of distinguishing without errors between the two states given by Eqs. (20) and (21), at the expense of allowing inconclusive results to occur where the procedure fails, is a special case of unambiguous quantum state filtering [12]. In the latter problem we want to discriminate, without error, a quantum state $|\psi\rangle$ occurring with the prior probability $\eta_1$, from a set of states $\{|\psi_j\rangle\}$, with prior probabilities $\eta_j'$. With the substitution $\eta_j' = \eta_2/d$ $(j=1,\ldots,d)$, the solution given in Ref. [12] yields for our specific example the minimum failure probability

$$Q_F = \begin{cases} 2\sqrt{\eta_1\frac{\eta_2}{d}}\|\psi^{\parallel}\| & \text{if} \quad \|\psi^{\parallel}\|^4 < \frac{\eta_2}{\eta_1 d}\|\psi^{\parallel}\|^2 < 1, \\ \eta_1 + \frac{\eta_2}{d}\|\psi^{\parallel}\|^2 & \text{if} \quad \frac{\eta_2}{\eta_1 d}\|\psi^{\parallel}\|^2 \geqslant 1, \\ \eta_1\|\psi^{\parallel}\|^2 + \frac{\eta_2}{d} & \text{if} \quad \frac{\eta_2}{\eta_1 d} \leqslant \|\psi^{\parallel}\|^2, \end{cases} \tag{28}$$

where $\|\psi^{\parallel}\|$ is defined by Eq. (23), and where again $\eta_1 + \eta_2 = 1$. As shown in Ref. [12], the second and the third line of Eq. (28) refer to two different types of von Neumann measurements while the failure probability given in the first line can be reached only by a generalized measurement. In the following we compare the minimum-error probability $P_E$ with the minimum failure probability $Q_F$, considering several special cases.

A considerable simplification arises when the states $|\psi\rangle, |u_1\rangle, \ldots, |u_d\rangle$ all occur with the same prior probabilities, i.e., when $\eta_1 = \eta_2/d = 1/(d+1)$ [23]. In this case Eq. (27) yields

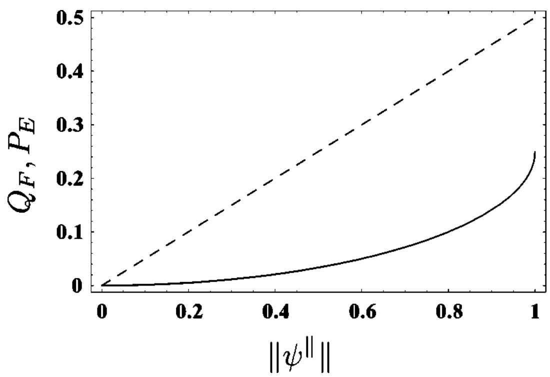


FIG. 1. Minimum probabilities of error in ambiguous discrimination $P_E$ (full line), and of failure in unambiguous discrimination $Q_F$ (dashed line), for distinguishing between $\rho_1 = |\psi\rangle\langle\psi|$ and $\rho_2 = \frac{1}{3}\Sigma_{j=1}^{3}|u_j\rangle\langle u_j|$, where $\langle u_i|u_j\rangle = \delta_{ij}$. The probabilities are plotted vs the norm of the parallel component, $\|\psi^{\|}\| = (\Sigma_{j=1}^{3}|\langle u_j|\psi\rangle|^2)^{1/2}$, and the prior probability of $\rho_1$ is assumed to be $\eta_1 = 0.25$.

$$P_E = \frac{1}{d+1}\left(1 - \sqrt{1 - \|\psi^{\|}\|^2}\right), \tag{29}$$

and from Eq. (28) we obtain

$$Q_F = \frac{2}{d+1}\|\psi^{\|}\|, \tag{30}$$

where the latter expression is valid in the whole range of the possible values of $\|\psi^{\|}\|$. When $\rho_1$ and $\rho_2$ are nearly orthogonal, i.e., when $\|\psi^{\|}\| \ll 1$, the minimum-error probability $P_E$ takes the approximate value $\|\psi^{\|}\|^2/(2d+2)$ and is therefore significantly smaller than the minimum failure probability $Q_F$ that is achievable in unambiguous discrimination (see Fig. 1). On the other hand, when $\|\psi^{\|}\| = 1$ the ratio $Q_F/P_E = 2$ is reached. This is an example from which it becomes obvious that the bound given by the general inequality Eq. (19) is tight.

For arbitrary prior probabilities we first investigate the discrimination in the linearly dependent case $\|\psi^{\|}\| = 1$ (see Fig. 2). From Eq. (27) with $\eta_2 = 1 - \eta_1$ we then find the

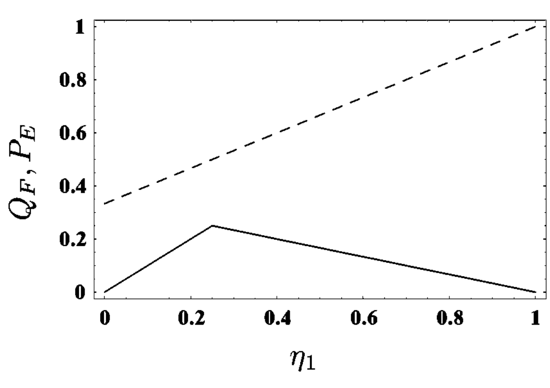


FIG. 2. Minimum-error probability in ambiguous discrimination $P_E$ (full line) and minimum failure probability in unambigous discrimination $Q_F$ (dashed line) for the states $\rho_1$ and $\rho_2$ specified in Fig. 1. The probabilities are depicted vs the prior probability $\eta_1$ of the state $\rho_1$, in the special case where $\|\psi^{\|}\| = 1$.

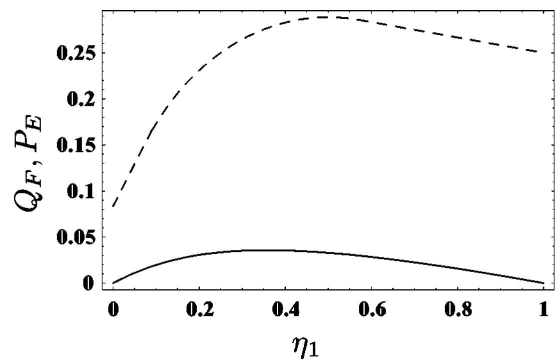


FIG. 3. Same as Fig. 2 but for the case $\|\psi^{\|}\| = 0.5$.

minimum-error probability

$$P_E = \frac{1}{2d}[1 + \eta_1(d-1) - |1 - \eta_1(d+1)|]. \tag{31}$$

Hence as long as $\eta_1 \leqslant 1/(d+1)$, which is equivalent to $\eta_1 \leqslant \eta_2/d$, we get $P_E = \eta_1$. As discussed in connection with the eigenvalues given in Eq. (26), the best discrimination strategy is then to always guess the quantum system to be in the state $\rho_2$, and it is not necessary to perform any measurement at all. However, for $\eta_1 \geqslant 1/(d+1)$ and $\|\psi^{\|}\| = 1$ we obtain the minimum-error probability $P_E = (1-\eta_1)/d$. Now the optimum strategy for minimum-error discrimination is to infer the system to be in the state $\rho_1$ when the detector along $|\psi\rangle$ clicks, which is just the eigenstate belonging to the negative eigenvalue, and that the state is $\rho_2$ for a click in any projection onto a direction orthogonal to $|\psi\rangle$. With $\|\psi^{\|}\| = 1$, from Eq. (28) the minimum failure probability in unambiguous discrimination follows to be

$$Q_F = \eta_1 + \frac{1}{d}(1 - \eta_1). \tag{32}$$

The strategy for optimum unambiguous discrimination in this case is also the von Neumann measurement consisting of projections onto the state $|\psi\rangle$ and onto the subspace orthogonal to $|\psi\rangle$. When a click occurs from projection onto the orthogonal subspace, the state $\rho_2$ is uniquely identified. The measurement fails to give a conclusive answer when either the state $|\psi\rangle$ was present, which occurs with probability $\eta_1$, or when the state $\rho_2$ was present and a click resulted from projection onto $|\psi\rangle$, which occurs with the probability $\eta_2/d$.

Finally in Fig. 3 an example is depicted for arbitrary prior probabilities and linearly independent states, where $\|\psi^{\|}\| < 1$. Obviously the minimum-error probability $P_E$ given by Eq. (27) is in general much smaller than the minimum failure probability $Q_F/2$ given by Eq. (28).

## IV. CONCLUSIONS

We showed that the minimum-error probability $P_E$ for ambiguously distinguishing any two mixed quantum states without inconclusive results is always at most half as large as the minimum failure probability $Q_F$ for unambiguous, i.e., error-free discrimination of the same two states, at the expense of the occurrence of inconclusive results where the measurement fails. As an example, we gave an exact analytical solution to the problem of determining whether the state of the quantum system is either a given pure state occurring with arbitrary prior probability, or a uniform statistical mixture of any number of mutually orthogonal states. Uniformly, i.e., completely mixed states have been considered in the context of estimating the quality of a source of quantum states, as has been recently discussed in connection with single-photon sources, introducing the new measure of suitability [29]. This measure relies on identifying all states that would be useful for the specific application, finding a set of states spanning the space of the useful states, and then defining a target state as a complete mixture of those states. If the mutually orthogonal states in the uniform statisitical mixture span the entire state space of the quantum system, the mixed state describes a totally random state, containing no information at all. Discriminating between the pure state and the mixed state then amounts to deciding whether the system has been reliably prepared in the pure state, or whether the preparation has failed [27].

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